University of New Brunswick
Faculty of Computer Science
CS1303: Discrete Structures
Homework Assignment 4, ***Due Time, Date*** *11:59 PM, March 9, 2021*

Student Name: ____________________________ Matriculation Number: ________________

Instructor: Rongxing Lu
The marking scheme is shown in the left margin and [100] constitutes full marks.

[**10**] 1. Write a negation for the given statement, and use a counterexample to disprove the given statement. Explain how the counterexample actually shows that the given statement is false.

(a) For all real numbers a and b, if $a < b$ then $a^2 < b^2$.

(b) For every integer n, if n is odd then $\frac{n-1}{2}$ is odd.

[**15**] 2. Disprove each of the following statements by giving a counterexample. In each case explain how the counterexample actually disproves the statement.

(a) For all integers m and n, if $2m + n$ is odd then m and n are both odd.

(b) For every integer p, if p is prime then $p^2 - 1$ is even.

(c) For every integer n, if n is even then $n^2 + 1$ is prime.

[**20**] 3. Determine which of the following statements are true and which are false. Prove each true statement directly from the definitions, and give a counterexample for each false statement.

(a) The product of any two rational numbers is a rational number.

(b) The quotient of any two rational numbers is a rational number.

(c) The difference of any two rational numbers is a rational number.

(d) If r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.

[**25**] 4. Determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

(a) The sum of any three consecutive integers is divisible by 3.

(b) The product of any two even integers is a multiple of 4.

(c) For all integers a, b, and c, if $a|b$ and $a|c$ then $a|(2b - 3c)$.

(d) For all integers a, b, and c, if $ab|c$ then $a|c$ and $b|c$.

(e) For all integers a, b, and c, if $a|(b + c)$ then $a|b$ or $a|c$.

[**30**] 5. Prove each of the following statements by contradiction. In addition, for (e) and (f), please also prove them by contraposition.

(a) For all odd integers a and b, $b^2 - a^2 \neq 4$.

(b) For all prime numbers a, b, and c, $a^2 + b^2 \neq c^2$.

(c) If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

(d) For any integer n, $n^2 - 2$ is not divisible by 4.

(e) The negative of any irrational number is irrational.

(f) For every integer n, if n^2 is odd then n is odd.

Solutions.

[**10**] 1. Write a negation for the given statement, and use a counterexample to disprove the given statement. Explain how the counterexample actually shows that the given statement is false.

(a) For all real numbers a and b, if $a < b$ then $a^2 < b^2$.

✓

$$\forall a, b \in R, (a < b) \to (a^2 < b^2)$$

Negation

$$\begin{aligned} & \neg(\forall a, b \in R, (a < b) \to (a^2 < b^2)) \\ \equiv\ & \exists a, b \in R, \neg((a < b) \to (a^2 < b^2))) \\ \equiv\ & \exists a, b \in R, \neg(\neg(a < b) \vee (a^2 < b^2))) \\ \equiv\ & \exists a, b \in R, \neg\neg(a < b) \wedge \neg(a^2 < b^2) \\ \equiv\ & \exists a, b \in R, (a < b) \wedge (a^2 \geq b^2) \end{aligned}$$

Counterexample. When $a = -5$, $b = 2$, we have $a < b$, but $a^2 = 25 > b^2 = 4$.

Because of the counterexample, we can disprove $\forall a, b \in R, (a < b) \to (a^2 < b^2)$.

(b) For every integer n, if n is odd then $\frac{n-1}{2}$ is odd.

✓

$$\forall n \in Z, Odd(n) \to Odd(\frac{n-1}{2})$$

Negation

$$\begin{aligned} & \neg(\forall n \in Z, Odd(n) \to Odd(\tfrac{n-1}{2})) \\ \equiv\ & \exists n \in Z, \neg(Odd(n) \to Odd(\tfrac{n-1}{2})) \\ \equiv\ & \exists n \in Z, \neg(\neg Odd(n) \vee Odd(\tfrac{n-1}{2})) \\ \equiv\ & \exists n \in Z, \neg\neg Odd(n) \wedge \neg Odd(\tfrac{n-1}{2}) \\ \equiv\ & \exists n \in Z, Odd(n) \wedge Even(\tfrac{n-1}{2}) \end{aligned}$$

Counterexample. When $n = 5$, $\frac{n-1}{2} = \frac{5-1}{2} = 2$, we have $Odd(n) = Odd(5)$ and $Even(\frac{n-1}{2}) = Even(2)$.

Because of the counterexample, we can disprove $\forall n \in Z, Odd(n) \to Odd(\frac{n-1}{2})$.

[**15**] 2. Disprove each of the following statements by giving a counterexample. In each case explain how the counterexample actually disproves the statement.

(a) For all integers m and n, if $2m + n$ is odd then m and n are both odd.

✓

$$\forall m, n \in Z, Odd(2m + n) \to Odd(m) \wedge Odd(n)$$

Negation

$$
\begin{aligned}
& \neg(\forall m, n \in Z, Odd(2m+n) \rightarrow Odd(m) \land Odd(n)) \\
\equiv\ & \exists m, n \in Z, \neg(Odd(2m+n) \rightarrow Odd(m) \land Odd(n)) \\
\equiv\ & \exists m, n \in Z, \neg(\neg Odd(2m+n) \lor (Odd(m) \land Odd(n))) \\
\equiv\ & \exists m, n \in Z, \neg\neg Odd(2m+n) \land \neg(Odd(m) \land Odd(n)) \\
\equiv\ & \exists m, n \in Z, Odd(2m+n) \land \neg(Odd(m) \land Odd(n)) \\
\equiv\ & \exists m, n \in Z, Odd(2m+n) \land (\neg Odd(m) \lor \neg Odd(n)) \\
\equiv\ & \exists m, n \in Z, Odd(2m+n) \land (Even(m) \lor Even(n))
\end{aligned}
$$

Counterexample. When $m = 2, n = 1$, we have $Odd(2m+n) = Odd(2*2+1) = Odd(5)$ and $Even(m) \lor Even(n) = Even(2) \lor Even(1)$ is true.

Because of the counterexample, we can disprove $\forall m, n \in Z, Odd(2m+n) \rightarrow Odd(m) \land Odd(n)$.

(b) For every integer p, if p is prime then $p^2 - 1$ is even.

✓

$$\forall p \in Z, Prime(p) \rightarrow Even(p^2 - 1)$$

Negation

$$
\begin{aligned}
& \neg(\forall p \in Z, Prime(p) \rightarrow Even(p^2 - 1)) \\
\equiv\ & \exists p \in Z, \neg(Prime(p) \rightarrow Even(p^2 - 1)) \\
\equiv\ & \exists p \in Z, \neg(\neg Prime(p) \lor Even(p^2 - 1)) \\
\equiv\ & \exists p \in Z, \neg\neg Prime(p) \land \neg Even(p^2 - 1) \\
\equiv\ & \exists p \in Z, Prime(p) \land Odd(p^2 - 1)
\end{aligned}
$$

Counterexample. When $p = 2$, we have $Prime(p) = Prime(2)$ and $Odd(p^2 - 1) = Odd(2^2 - 1) = Odd(3)$ is true.

Because of the counterexample, we can disprove $\forall p \in Z, Prime(p) \rightarrow Even(p^2 - 1)$.

(c) For every integer n, if n is even then $n^2 + 1$ is prime.

✓

$$\forall n \in Z, Even(n) \rightarrow Prime(n^2 + 1)$$

Negation

$$
\begin{aligned}
& \neg(\forall n \in Z, Even(n) \rightarrow Prime(n^2 + 1)) \\
\equiv\ & \neg(\forall n \in Z, Even(n) \rightarrow Prime(n^2 + 1)) \\
\equiv\ & \exists n \in Z, \neg(Even(n) \rightarrow Prime(n^2 + 1)) \\
\equiv\ & \exists n \in Z, \neg(\neg Even(n) \lor Prime(n^2 + 1)) \\
\equiv\ & \exists n \in Z, \neg\neg Even(n) \land \neg Prime(n^2 + 1) \\
\equiv\ & \exists n \in Z, Even(n) \land \neg Prime(n^2 + 1)
\end{aligned}
$$

Counterexample. When $n = 0$, we have $Even(n) = Even(0)$ and $\neg Prime(n^2+1) = \neg Prime(0^2+1) = \neg Prime(1)$ is true.

Because of the counterexample, we can disprove $\forall n \in Z, Even(n) \rightarrow Prime(n^2 + 1)$.

[**20**] 3. Determine which of the following statements are true and which are false. Prove each true statement directly from the definitions, and give a counterexample for each false statement.

(a) The product of any two rational numbers is a rational number.

$\checkmark$

Define $Q(x)$ as x is a rational number

$$\forall x, y \in R, Q(x) \wedge Q(y) \rightarrow Q(x \cdot y)$$

Proof. As $\forall x, y \in R, Q(x) \wedge Q(y)$, based on the definition of rational, we have
$x = \frac{a}{b}$, where $a, b \in Z$, and $b \neq 0$
$y = \frac{c}{d}$, where $c, d \in Z$, and $d \neq 0$
Then,

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Since the multiplication is closed in integer, $a, c \in Z$, we have $a \cdot c \in Z$; $b, d \in Z$, we have $b \cdot d \in Z$;
$b \neq 0, d \neq 0$, we have $b \cdot d \neq 0$ from the zero product property.
Therefore, based on the definition of rational, we have

$$\forall x, y \in R, Q(x) \wedge Q(y) \rightarrow Q(x \cdot y)$$

That is, the statement is true.

(b) The quotient of any two rational numbers is a rational number.

$\checkmark$

Define $Q(x)$ as x is a rational number

$$\forall x, y \in R, Q(x) \wedge Q(y) \rightarrow Q(\frac{x}{y})$$

Proof. As $\forall x, y \in R, Q(x) \wedge Q(y)$, based on the definition of rational, we have
$x = \frac{a}{b}$, where $a, b \in Z$, and $b \neq 0$
$y = \frac{c}{d}$, where $c, d \in Z$, and $d \neq 0$
Then,

$$\frac{x}{y} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \cdot d}{b \cdot c}$$

Since the multiplication is closed in integer, $a, d \in Z$, we have $a \cdot d \in Z$; $b, c \in Z$, we have $b \cdot c \in Z$;
However, when $c = 0$, we will have $b \cdot c = 0$.
Counterexample. $a = 2, b = 1$, $x = \frac{a}{b} = \frac{2}{1} = 2$; $c = 0, d = 1$, $y = \frac{c}{d} = \frac{0}{1} = 0$. Then, $\frac{x}{y} = \frac{2}{0}$ is not a rational number.
Therefore, we disprove

$$\forall x, y \in R, Q(x) \wedge Q(y) \rightarrow Q(\frac{x}{y})$$

That is, the statement is false.

(c) The difference of any two rational numbers is a rational number.

✓

Define $Q(x)$ as x is a rational number

$$\forall x, y \in R, Q(x) \wedge Q(y) \rightarrow Q(x-y)$$

Proof. As $\forall x, y \in R, Q(x) \wedge Q(y)$, based on the definition of rational, we have
$x = \frac{a}{b}$, where $a, b \in Z$, and $b \neq 0$
$y = \frac{c}{d}$, where $c, d \in Z$, and $d \neq 0$
Then,

$$x - y = \frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - c \cdot b}{b \cdot d}$$

Since the multiplication and substraction are closed in integer, $a, b, c, d \in Z$, we have $a \cdot d - c \cdot b \in Z$;
$b, d \in Z$, we have $b \cdot d \in Z$;
$b \neq 0, d \neq 0$, we have $b \cdot d \neq 0$ from the zero product property.
Therefore, based on the definition o rational, we have

$$\forall x, y \in R, Q(x) \wedge Q(y) \rightarrow Q(x-y)$$

That is, the statement is true.

(d) If r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.

✓

Define $Q(x)$ as x is a rational number

$$\forall r, s \in R, Q(r) \wedge Q(s) \rightarrow Q(\frac{r+s}{2})$$

Proof. As $\forall r, s \in R, Q(r) \wedge Q(s)$, based on the definition of rational, we have
$r = \frac{a}{b}$, where $a, b \in Z$, and $b \neq 0$
$s = \frac{c}{d}$, where $c, d \in Z$, and $d \neq 0$
Then,

$$\frac{r+s}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{a \cdot d + c \cdot b}{2b \cdot d}$$

Since the multiplication and addition are closed in integer, $a, b, c, d \in Z$, we have $a \cdot d + c \cdot b \in Z$;
$b, d \in Z$, we have $2b \cdot d \in Z$;
$b \neq 0, d \neq 0$, we have $2b \cdot d \neq 0$ from the zero product property.
Therefore, based on the definition o rational, we have

$$\forall r, s \in R, Q(r) \wedge Q(s) \rightarrow Q(\frac{r+s}{2})$$

That is, the statement is true.

[**25**] 4. Determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

(a) The sum of any three consecutive integers is divisible by 3.

✓

Denote $a|b$ as b can be divided by a for $a, b \in Z$, i.e., $b = a \cdot k$ for some $k \in Z$.

$$\forall x \in Z, 3|(x + (x+1) + (x+2))$$

Proof. $\forall x \in Z$, three consecutive integers can be written as x, $x+1$, and $x+2$. Then, their sum

$$s = x + (x+1) + (x+2) = 3x + 3 = 3(x+1)$$

Based on the definition of $a|b$, we have $3|s$. As a result, we prove

$$\forall x \in Z, 3|(x + (x+1) + (x+2))$$

That is, the sum of any three consecutive integers is divisible by 3.

(b) The product of any two even integers is a multiple of 4.

✓

$$\forall x, y \in Z, Even(x) \wedge Even(y) \rightarrow \exists m \in Z, x \cdot y = 4 \cdot m$$

Prof. As $\forall x, y \in Z, Even(x) \wedge Even(y)$, based on the definition of even, we have
$x = 2k$ for some $k \in Z$, and $y = 2q$ for some $q \in Z$. Then,

$$x \cdot y = 2k \cdot 2q = 4k \cdot q$$

Set $m = k \cdot q$, as $k, q \in Z$, we have $m = k \cdot q \in Z$. Then, we have $x \cdot y = 4 \cdot m$. As a result, we prove

$$\forall x, y \in Z, Even(x) \wedge Even(y) \rightarrow \exists m \in Z, x \cdot y = 4 \cdot m$$

That is, the product of any two even integers is a multiple of 4.

(c) For all integers a, b, and c, if $a|b$ and $a|c$ then $a|(2b - 3c)$.

✓

Denote $a|b$ as b can be divided by a for $a, b \in Z$, i.e., $b = a \cdot k$ for some $k \in Z$.

$$\forall a, b, c \in Z, (a|b) \wedge (a|c) \rightarrow a|(2b - 3c)$$

Proof. For all integers a, b, and c, based on the definition of $a|b$ and $a|c$, we have
$b = a \cdot k$ for some $k \in Z$, $c = a \cdot q$ for some $q \in Z$.
From $b = a \cdot k$, we have $2b = a \cdot 2k$.
From $c = a \cdot q$, we have $-3c = a \cdot (-3)q$. Then,

$$2b - 3c = a \cdot 2k + a \cdot (-3q) = a \cdot (2k - 3q)$$

Let $m = 2k-3q$, as $k, q \in Z$, as the multiplication and subtract are close in integer, $m = 2k-3q \in Z$. Based on the definition of $a|b$, we prove

$$\forall a, b, c \in Z, (a|b) \wedge (a|c) \rightarrow a|(2b-3c)$$

That is, for all integers a, b, and c, if $a|b$ and $a|c$ then $a|(2b-3c)$.

(d) For all integers a, b, and c, if $ab|c$ then $a|c$ and $b|c$.

✓

Denote $a|b$ as b can be divided by a for $a, b \in Z$, i.e., $b = a \cdot k$ for some $k \in Z$.

$$\forall a, b, c \in Z, ab|c \rightarrow (a|c) \wedge (b|c)$$

Proof. for $\forall a, b, c \in Z$, from the definition of $ab|c$, we know
$c = ab \cdot k$ for some $k \in Z$.
Let $k_1 = b \cdot k$, we know $k_1 \in Z$, so we have $c = a \cdot k_1$, and thus we have $a|c$.
Let $k_2 = a \cdot k$, we know $k_2 \in Z$, so we have $c = b \cdot k_2$, and thus we have $b|c$.
By combining them, we have

$$\forall a, b, c \in Z, ab|c \rightarrow (a|c) \wedge (b|c)$$

Therefore, for all integers a, b, and c, if $ab|c$ then $a|c$ and $b|c$.

(e) For all integers a, b, and c, if $a|(b+c)$ then $a|b$ or $a|c$.

✓

Denote $a|b$ as b can be divided by a for $a, b \in Z$, i.e., $b = a \cdot k$ for some $k \in Z$.

$$\forall a, b, c \in Z, a|(b+c) \rightarrow (a|b) \vee (a|c)$$

The negation is

$$\begin{aligned}
& \neg(\forall a, b, c \in Z, a|(b+c) \rightarrow (a|b) \vee (a|c)) \\
\equiv\ & \exists a, b, c \in Z, \neg(a|(b+c) \rightarrow (a|b) \vee (a|c)) \\
\equiv\ & \exists a, b, c \in Z, \neg(\neg(a|(b+c)) \vee ((a|b) \vee (a|c))) \\
\equiv\ & \exists a, b, c \in Z, \neg\neg(a|(b+c)) \wedge \neg((a|b) \vee (a|c))) \\
\equiv\ & \exists a, b, c \in Z, (a|(b+c)) \wedge \neg((a|b) \vee (a|c))) \\
\equiv\ & \exists a, b, c \in Z, (a|(b+c)) \wedge (\neg(a|b) \wedge \neg(a|c)) \\
\equiv\ & \exists a, b, c \in Z, a|(b+c) \wedge a \nmid b \wedge a \nmid c
\end{aligned}$$

Counterexample. When $a = 10$, $b = 4$, $c = 6$, we have $a|(b+c)$, $a \nmid b$, and $a \nmid c$. Therefore, this counterexample disprove

$$\forall a, b, c \in Z, a|(b+c) \rightarrow (a|b) \vee (a|c)$$

That is, the statement "For all integers a, b, and c, if $a|(b+c)$ then $a|b$ or $a|c$" is false.

[**30**] 5. Prove each of the following statements by contradiction. In addition, for (e) and (f), please also prove them by contraposition.

(a) For all odd integers a and b, $b^2 - a^2 \neq 4$.

✓

$$P: \qquad \forall a, b \in Z, Odd(a) \wedge Odd(b) \rightarrow (b^2 - a^2 \neq 4)$$

Negation:

$$\neg P: \qquad \exists a, b \in Z, Odd(a) \wedge Odd(b) \wedge (b^2 - a^2 = 4)$$

Proof by contradiction.

Suppose $\neg P$ is true. From $b^2 - a^2 = 4$, we know $(b + a)(b - a) = 4$.

Since $a, b \in Z$, $x = b + a$, $y = b - a$ are also integers. Then, there are only 6 cases that $x \cdot y = 4$.

	x	y	$x \cdot y = 4$
case 1	1	4	4
case 2	4	1	4
case 3	2	2	4
case 4	-1	-4	4
case 5	-4	-1	4
case 6	-2	-2	4

- Case 1: $b + a = 4, b - a = 1$. In this case, we have $2b = 5 \Rightarrow b = \frac{5}{2} \notin Z$, which of course means $\neg Odd(b)$. Then, $\neg Odd(b) \wedge Odd(b) = \mathbf{c}$
- Case 2: $b + a = 1, b - a = 4$. In this case, we have $2b = 5 \Rightarrow b = \frac{5}{2} \notin Z$, which of course means $\neg Odd(b)$. Then, $\neg Odd(b) \wedge Odd(b) = \mathbf{c}$
- Case 3: $b + a = 2, b - a = 2$. In this case, we have $b = 2 \in Z$, which means $Even(b) = \neg Odd(b)$. Then, $\neg Odd(b) \wedge Odd(b) = \mathbf{c}$.
- Case 4: $b + a = -4, b - a = -1$. In this case, we have $2b = -5 \Rightarrow b = \frac{-5}{2} \notin Z$, which of course means $\neg Odd(b)$. Then, $\neg Odd(b) \wedge Odd(b) = \mathbf{c}$
- Case 5: $b + a = -1, b - a = -4$. In this case, we have $2b = -5 \Rightarrow b = \frac{-5}{2} \notin Z$, which of course means $\neg Odd(b)$. Then, $\neg Odd(b) \wedge Odd(b) = \mathbf{c}$
- Case 6: $b + a = -2, b - a = -2$. In this case, we have $b = -2 \in Z$, which means $Even(b) = \neg Odd(b)$. Then, $\neg Odd(b) \wedge Odd(b) = \mathbf{c}$.

Therefore, the supposition "$\neg P$ is true" is false, and P is true. That is, for all odd integers a and b, $b^2 - a^2 \neq 4$.

(b) For all prime numbers a, b, and c, $a^2 + b^2 \neq c^2$.

✓

$$P: \qquad \forall a, b, c \in Z, \textit{Prime}(a) \wedge \textit{Prime}(b) \wedge \textit{Prime}(c) \rightarrow (a^2 + b^2 \neq c^2)$$

Negation

$$\neg P: \qquad \exists a, b, c \in Z, \textit{Prime}(a) \wedge \textit{Prime}(b) \wedge \textit{Prime}(c) \wedge (a^2 + b^2 = c^2)$$

Proof by contradiction.

Suppose $\neg P$ is true. From $a^2 + b^2 = c^2$, we know $a^2 = c^2 - b^2$, i.e., $a^2 = (c - b) \cdot (c + b)$.

Because *Prime*$(a) \wedge$ *Prime*$(b) \wedge$ *Prime*(c) and the smallest prime number is 2, then, $a \geq 2, b \geq 2, c \geq 2$, and it is easy to see $a^2 \geq 4$, $c + b \geq 4$. Then, from $c - b = \frac{a^2}{c+b}$, we know $c - b$ should also be a positive integer.

Let $x = c + b$, $y = c - b$, we have $x, y \in Z^+$. Then, since $Prime(a)$, there are three cases for $a^2 = x \cdot y$.

	x	y	$a^2 = x \cdot y$
case 1	a^2	1	a^2
case 2	1	a^2	a^2
case 3	a	a	a^2

- Case 1: $c + b = a^2, c - b = 1$. In this case, we have $2b = (a^2 - 1) \Rightarrow b = \frac{a^2-1}{2}$.
  - 1.1: if the prime $a = 2$, $b = \frac{a^2-1}{2} = \frac{3}{2} \notin Z$, which of course shows $\neg Prime(b)$. Then, $\neg Prime(b) \wedge Prime(b) =$ **c**
  - 1.2: if the prime $a > 2$, a is odd, i.e., $a = 2k + 1$ for some $k \in Z$ and $k > 1$. Then, $b = \frac{a^2-1}{2} = \frac{(2k+1)^2-1}{2} = \frac{4k^2+4k+1-1}{2} = \frac{4k^2+4k}{2} = 2k^2 + 2k$. Let $m = k^2 + k$, we know $m > 2$ and $m \in Z$, thus from $b = 2m$, we know $Even(b) \wedge (b > 4)$. From $Even(b) \wedge (b > 4)$, we know $\neg Prime(b)$. Then, $\neg Prime(b) \wedge Prime(b) =$ **c**.
- Case 2: $c + b = 1, c - b = a^2$. In this case, we have $2b = (1 - a^2) \Rightarrow b = \frac{1-a^2}{2}$. Since *Prime*$(a)$, i.e., $a \geq 2$, we know $a^2 \geq 4$ and $-a^2 \leq -4$. Then, $b = \frac{1-a^2}{2} \leq \frac{1-4}{2} = \frac{-3}{2}$. Then, b is negative, which of course means $\neg Prime(b)$. Then, $\neg Prime(b) \wedge Prime(b) =$ **c**.
- Case 3: $c + b = a, c - b = a$. In this case, we have $b = 0 \in Z$, which means $\neg Prime(b)$. Then, $\neg Prime(b) \wedge Prime(b) =$ **c**.

Therefore, the supposition "$\neg P$ is true" is false, and P is true. That is, for all prime numbers a, b, and c, $a^2 + b^2 \neq c^2$.

(c) If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

✓

Define $Q(x)$ as x is a rational number

$$P: \qquad \forall a, b, r \in R, Q(a) \wedge Q(b) \wedge (b \neq 0) \wedge \neg Q(r) \rightarrow \neg Q(a + br)$$

Negation

$$\neg P: \qquad \exists a, b, r \in R, Q(a) \wedge Q(b) \wedge (b \neq 0) \wedge \neg Q(r) \wedge Q(a + br)$$

Proof by contradiction.

Suppose $\neg P$ is true. As $\exists a, b, r \in R, Q(a) \wedge Q(b) \wedge (b \neq 0) \wedge Q(a + br)$. Then, based on the definition of rational, we have

$a = \frac{u}{v}$ where $u, v \in Z$ and $v \neq 0$.

$b = \frac{s}{t}$ where $s, t \in Z$ and $t \neq 0$. Because $b \neq 0$, we have $s \neq 0$.

$a + br = \frac{x}{y}$ where $x, y \in Z$ and $y \neq 0$.

By substitution, we have $\frac{u}{v} + r \cdot \frac{s}{t} = \frac{x}{y}$, i.e., $r \cdot \frac{s}{t} = \frac{x}{y} - \frac{u}{v} = \frac{x \cdot v - y \cdot u}{y \cdot v}$. Then,

$$r = \frac{t(x \cdot v - y \cdot u)}{s \cdot y \cdot v}$$

Because $x, y, u, v, t \in Z$, we have $t(x \cdot v - y \cdot u) \in Z$. Also because $s, y, v \in Z$, we have $s \cdot y \cdot v \in Z$. As $s \neq 0, y \neq 0, v \neq 0$, we have $s \cdot y \cdot v \neq 0$. As a result, based on the definition of ration, we have $Q(r)$. Then, $\neg Q(r) \wedge Q(r) = \mathbf{c}$.

Therefore, the supposition "$\neg P$ is true" is false, and P is true. That is, if a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

(d) For any integer n, $n^2 - 2$ is not divisible by 4.

✓

Denote $a|b$ as $b \in Z$ is divisible by $a \in Z$, i.e, $b = a \cdot k$ for some $k \in Z$.

$$P: \qquad \forall n, n \in Z \to \neg(4|(n^2 - 2))$$

Negation

$$\neg P: \qquad \exists n, n \in Z \wedge 4|(n^2 - 2)$$

Proof by contradiction.

Suppose $\neg P$ is true. As $\exists n \in Z, 4|(n^2 - 2)$, based on the definition of $a|b$, we know $n^2 - 2 = 4 \cdot k$ for some $k \in Z$. Then, $n^2 = 4k + 2 = 2^2(\frac{2k+1}{2})$, and we have

$$n = \pm 2 \cdot \sqrt{\frac{2k+1}{2}}$$

As $2k + 1$ is odd, $\neg(2|(2k + 1))$, i.e, $2k + 1$ is not divisible by 2. Therefore, $n = \pm 2 \cdot \sqrt{\frac{2k+1}{2}}$ is not an integer, i.e., $n \notin Z$. Then, $(n \notin Z) \wedge (n \in Z) = \mathbf{c}$.

Therefore, the supposition "$\neg P$ is true" is false, and P is true. That is, for any integer n, $n^2 - 2$ is not divisible by 4.

(e) The negative of any irrational number is irrational.

✓

Define $Q(x)$ as x is an rational number

$$P: \qquad \forall x \in R, \neg Q(x) \to \neg Q(-x)$$

Negation

$$\neg P: \qquad \exists x \in R, \neg Q(x) \wedge Q(-x)$$

Contrapositive

$$CP: \qquad \forall x \in R, Q(-x) \to Q(x)$$

Proof by contradiction.

Suppose $\neg P$ is true. As $\exists x \in R, Q(-x)$, based on the definition of rational, we have

$-x = \frac{a}{b}$, for $a, b \in Z$ and $b \neq 0$. Then, by algebra, we have

$$x = -(-x) = -\frac{a}{b} = \frac{-a}{b}.$$

Because $a \in Z$, we have $-a \in Z$ from $-a = (-1) \cdot a$. Also, $b \in Z$ and $b \neq 0$, based on the definition of rational, we have x is a rational number, i.e., $Q(x)$. Then, $Q(x) \wedge \neg Q(x) = \mathbf{c}$.

Therefore, the supposition "$\neg P$ is true" is false, and P is true. That is, the negative of any irrational number is irrational.

Proof by contraposition.

$$CP: \qquad \forall x \in R, Q(-x) \rightarrow Q(x) \qquad \equiv P: \qquad \forall x \in R, \neg Q(x) \rightarrow \neg Q(-x)$$

As $\forall x \in R, Q(-x)$, based on the definition of rational, we have $-x = \frac{a}{b}$ for some $a, b \in Z$ and $b \neq 0$. Then, by algebra, we have

$$x = -(-x) = -\frac{a}{b} = \frac{-a}{b}.$$

Because $a \in Z$, we have $-a \in Z$ from $-a = (-1) \cdot a$. Also, $b \in Z$ and $b \neq 0$, based on the definition of rational, we have x is a rational number, i.e., $Q(x)$.

Because CP is true, and $CP \equiv P$. Therefore, P is also true. That is, the negative of any irrational number is irrational.

(f) For every integer n, if n^2 is odd then n is odd.

✓

$$P: \qquad \forall n \in Z, Odd(n^2) \rightarrow Odd(n).$$

Negation

$$\neg P: \qquad \exists n \in Z, Odd(n^2) \wedge \neg Odd(n).$$

Contrapositive

$$CP: \qquad \forall n \in Z, \neg Odd(n) \rightarrow \neg Odd(n^2).$$

Proof by contradiction.

Suppose $\neg P$ is true. As $\exists n \in Z, \neg Odd(n)$, we know $Even(n)$. Based on the definition of even, we have $n = 2k$ for some $k \in Z$. Then

$$n^2 = (2k)^2 = 4k^2 = 2 \times 2k^2.$$

Let $m = 2k^2$, it is easy to know $m \in Z$. Then $n^2 = 2m$, based on the definition of even, we know $Even(n^2)$. Then, $Even(n^2) \wedge Odd(n^2) = \mathbf{c}$.

Therefore, the supposition "$\neg P$ is true" is false, and P is true. That is, for every integer n, if n^2 is odd then n is odd.

Proof by contraposition.

$$CP: \qquad \forall n \in Z, \neg Odd(n) \to \neg Odd(n^2) \qquad \equiv P: \qquad \forall n \in Z, Odd(n^2) \to Odd(n).$$

As $\forall n \in Z, \neg Odd(n)$, we know $Even(n)$. Then, based on the definition of even, we have $n = 2k$ for some $k \in Z$.

Then,

$$n^2 = (2k)^2 = 4k^2 = 2 \times 2k^2.$$

Let $m = 2k^2$, it is easy to know $m \in Z$. Then $n^2 = 2m$, based on the definition of even, we know $Even(n^2)$.

Because CP is true, and $CP \equiv P$. Therefore, P is also true. That is, for every integer n, if n^2 is odd then n is odd.