University of New Brunswick
Faculty of Computer Science
CS1303: Discrete Structures
Homework Assignment 5, **Due Time, Date** *11:59 PM, March 23, 2021*

Student Name: ____________________________ Matriculation Number: ________________

Instructor: Rongxing Lu
The marking scheme is shown in the left margin and [100] constitutes full marks.

[**5**] 1. Transform each of the following by making the change of variable $j = i - 1$.

(a)
$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n}$$

(b)
$$\sum_{i=3}^{n} \frac{i}{i+n-1}$$

[**5**] 2. Write each of following as a single summation or product.

(a)
$$3 \cdot \sum_{k=1}^{n}(2k-3) + \sum_{k=1}^{n}(4-5k)$$

(b)
$$\left(\prod_{k=1}^{n} \frac{k}{k+1}\right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right)$$

[**10**] 3. Compute each of the following. Assume the values of the variables are restricted so that the expressions are defined.

(a)
$$\frac{4!}{3!}$$

(b)
$$\frac{3!}{0!}$$

(c)
$$\frac{(n-1)!}{(n+1)!}$$

(d)
$$\frac{n!}{(n-k+1)!}$$

[**28**] 4. Prove each of the following statements using mathematical induction.

(a) For every integer $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n-3)}{2}.$$

(b) For every integer $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}.$$

(c) For every integer $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(d) For every integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

[**32**] 5. Prove each of the following statements using mathematical induction.

(a) For each integer $n \geq 0$, $3^{2n} - 1$ is divisible by 8.

(b) For each integer $n \geq 2$, $2^n < (n+1)!$.

(c) For each integer $n \geq 0$, $1 + 3n \leq 4^n$.

(d) For every real number $x > -1$ and every integer $n \geq 2$, $1 + nx \leq (1+x)^n$.

[**20**] 6. Prove each of the following statements using strong mathematical induction.

(a) Suppose $b_1, b_2, b_3, \cdots$ is a sequence defined as follows:

$$b_1 = 4, \quad b_2 = 12, \qquad b_k = b_{k-2} + b_{k-1} \qquad \text{for each integer } k \geq 3$$

Prove that b_n is divisible by 4 for every integer $n \geq 1$.

(b) Suppose $f_0, f_1, f_2, \cdots$ is a sequence defined as follows:

$$f_0 = 5, \quad f_1 = 16, \qquad f_k = 7f_{k-1} - 10f_{k-2} \qquad \text{for each integer } k \geq 2$$

Prove that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for every integer $n \geq 0$.

Solutions.

[**5**] 1. Transform each of the following by making the change of variable $j = i - 1$.

(a)

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n}$$

✓

Since $1 \leq i \leq n+1$, and $j = i-1$, i.e., $i = j+1$, we have $1 \leq j+1 \leq n+1$. Then, we have $0 \leq j \leq n$. As a result,

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n} = \sum_{j=0}^{n} \frac{j^2}{(j+1) \cdot n}$$

(b)

$$\sum_{i=3}^{n} \frac{i}{i+n-1}$$

✓

Since $3 \leq i \leq n$, and $j = i-1$, i.e., $i = j+1$, we have $3 \leq j+1 \leq n$. Then, we have $2 \leq j \leq n-1$. As a result,

$$\sum_{i=3}^{n} \frac{i}{i+n-1} = \sum_{j=2}^{n-1} \frac{j+1}{j+1+n-1} = \sum_{j=2}^{n-1} \frac{j+1}{j+n}$$

[**5**] 2. Write each of following as a single summation or product.

(a)

$$3 \cdot \sum_{k=1}^{n} (2k-3) + \sum_{k=1}^{n} (4-5k)$$

✓

$$3 \cdot \sum_{k=1}^{n} (2k-3) + \sum_{k=1}^{n} (4-5k) = \sum_{k=1}^{n} (3 \cdot (2k-3) + (4-5k)) = \sum_{k=1}^{n} (k-5)$$

(b)

$$\left(\prod_{k=1}^{n} \frac{k}{k+1} \right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2} \right)$$

✓

$$\left(\prod_{k=1}^{n} \frac{k}{k+1} \right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2} \right) = \prod_{k=1}^{n} \left(\frac{k}{k+1} \cdot \frac{k+1}{k+2} \right) = \prod_{k=1}^{n} \frac{k}{k+2}$$

[**10**] 3. Compute each of the following. Assume the values of the variables are restricted so that the expressions are defined.

(a)
$$\frac{4!}{3!}$$

✓

$$\frac{4!}{3!} = \frac{4 \cdot 3!}{3!} = 4$$

(b)
$$\frac{3!}{0!}$$

✓

$$\frac{3!}{0!} = \frac{3 \times 2 \times 1}{1} = 6$$

(c)
$$\frac{(n-1)!}{(n+1)!}$$

✓

$$\frac{(n-1)!}{(n+1)!} = \frac{(n-1)!}{(n+1) \cdot n \cdot (n-1)!} = \frac{1}{(n+1) \cdot n} = \frac{(n+1) - n}{(n+1) \cdot n} = \frac{1}{n} - \frac{1}{n+1}$$

(d)
$$\frac{n!}{(n-k+1)!}$$

✓ Based on the definition of factorial, we need $n - k + 1 \geq 0$, that is, $k \leq n + 1$.

$$\frac{n!}{(n-k+1)!} = \frac{n \cdot (n-1) \cdots (n-k+3) \cdot (n-k+2) \cdot (n-k+1)!}{(n-k+1)!} = \prod_{j=0}^{k-2} (n-j)$$

[**28**] 4. Prove each of the following statements using mathematical induction.

(a) For every integer $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n-4) = \frac{n(5n-3)}{2}.$$

✓

Proof by Mathematical Induction.

First, we can also write $1 + 6 + 11 + 16 + \cdots + (5n-4) = \frac{n(5n-3)}{2}$ as

$$\sum_{i=1}^{n}(5i-4) = \frac{n(5n-3)}{2}$$

Base Case: When $n = 1$,
Left-Hand Side (LHS):

$$LHS = \sum_{i=1}^{n}(5i-4) = \sum_{i=1}^{1}(5i-4) = 5 \cdot 1 - 4 = 1$$

Right-Hand Side (RHS):

$$RHS = \frac{n(5n-3)}{2} = \frac{1 \cdot (5 \cdot 1 - 3)}{2} = 1$$

Therefore, LHS = RHS.
Induction Step.
Assume when $n = k$, we have

$$\sum_{i=1}^{k}(5i-4) = \frac{k(5k-3)}{2}$$

Then, when $n = k+1$,

$$LHS = \sum_{i=1}^{n}(5i-4) = \sum_{i=1}^{k+1}(5i-4) = \underbrace{\sum_{i=1}^{k}(5i-4)}_{\text{using the assumption}} + 5(k+1) - 4$$

$$= \frac{k(5k-3)}{2} + 5(k+1) - 4 = \frac{k(5k-3) + 10(k+1) - 8}{2} = \frac{5k^2 - 3k + 10k + 2}{2}$$

$$= \frac{5k^2 + 10k + 5 - 3k - 3}{2} = \frac{5(k+1)^2 - 3(k+1)}{2}$$

$$RHS = \frac{n(5n-3)}{2} = \frac{(k+1)(5(k+1)-3)}{2} = \frac{5(k+1)^2 - 3(k+1)}{2}$$

Therefore, when $n = k+1$, we also have LHS = RHS.
As a result, we prove, for every integer $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n-4) = \frac{n(5n-3)}{2}.$$

(b) For every integer $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}.$$

✓

Proof by Mathematical Induction.

First, we can also write $4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n-16)}{3}$ as

$$\sum_{i=3}^{n} 4^i = \frac{4(4^n - 16)}{3}$$

Base Case: When $n = 3$,

$$LHS = \sum_{i=3}^{n} 4^i = \sum_{i=3}^{3} 4^i = 4^3$$

$$RHS = \frac{4(4^n - 16)}{3} = \frac{4(4^3 - 16)}{3} = 4^3$$

Therefore, LHS = RHS.

Induction Step.

Assume when $n = k$, we have

$$\sum_{i=3}^{k} 4^i = \frac{4(4^k - 16)}{3}$$

Then, when $n = k + 1$,

$$LHS = \sum_{i=3}^{n} 4^i = \sum_{i=3}^{k+1} 4^i = \underbrace{\sum_{i=3}^{k} 4^i}_{\text{using the assumption}} + 4^{k+1}$$

$$= \frac{4(4^k - 16)}{3} + 4^{k+1} = \frac{4(4^k - 16) + 3 \cdot 4^{k+1}}{3} = \frac{4^{k+1} - 4 \cdot 16 + 3 \cdot 4^{k+1}}{3}$$

$$= \frac{4 \cdot 4^{k+1} - 4 \cdot 16}{3} = \frac{4(4^{k+1} - 16)}{3}$$

$$RHS = \frac{4(4^n - 16)}{3} = \frac{4(4^{k+1} - 16)}{3}$$

Therefore, when $n = k + 1$, we also have LHS = RHS.

As a result, we prove, for every integer $n \geq 3$,

$$\sum_{i=3}^{n} 4^i = \frac{4(4^n - 16)}{3}$$

(c) For every integer $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

✓

Proof by Mathematical Induction.

First, we can also write $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ as

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base Case: When $n = 1$,

$$LHS = \sum_{i=1}^{n} i^2 = \sum_{i=1}^{1} i^2 = 1$$

$$RHS = \frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6} = 1$$

Therefore, LHS = RHS.

Induction Step.

Assume when $n = k$, we have

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Then, when $n = k + 1$,

$$LHS = \sum_{i=1}^{n} i^2 = \sum_{i=1}^{k+1} i^2 = \underbrace{\sum_{i=1}^{k} i^2}_{\text{using the assumption}} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{(k+1)(2k^2+k)}{6} + \frac{(k+1)(6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$RHS = \frac{n(n+1)(2n+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore, when $n = k + 1$, we also have LHS = RHS.

As a result, we prove, for every integer $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(d) For every integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

✓

Proof by Mathematical Induction.

Base Case: When $n = 1$,

$$LHS = \sum_{i=1}^{n} i(i!) = \sum_{i=1}^{1} i(i!) = 1 \cdot (1!) = 1$$

$$RHS = (n+1)! - 1 = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$$

Therefore, LHS = RHS.

Induction Step.

Assume when $n = k$, we have

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1$$

Then, when $n = k + 1$,

$$LHS = \sum_{i=1}^{n} i(i!) = \sum_{i=1}^{k+1} i(i!) = \underbrace{\sum_{i=1}^{k} i(i!)}_{\text{using the assumption}} + (k+1)(k+1)!$$

$$= (k+1)! - 1 + (k+1)(k+1)! = (k+1)! \cdot (k+2) - 1 = (k+2)! - 1$$

$$RHS = (n+1)! - 1 = (k+2)! - 1$$

Therefore, when $n = k + 1$, we also have LHS = RHS.

As a result, we prove, for every integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

[32] 5. Prove each of the following statements using mathematical induction.

(a) For each integer $n \geq 0$, $3^{2n} - 1$ is divisible by 8.

✓

Proof by Mathematical Induction.

Let $a|b$ be b is divisible by a, i.e., $b = a \cdot k$ for some $k \in Z$.

Let $f(n) = 3^{2n} - 1$.

Base Case: When $n = 0$,

$$f(0) = f(n) = 3^{2n} - 1 = 3^{2 \cdot 0} - 1 = 0$$

Therefore, $8|f(0)$.
Induction Step.
Assume when $n = k$, we have

$$8|f(k) \qquad i.e., \quad 8|(3^{2k} - 1), \quad f(k) = (3^{2k} - 1) = 8 \cdot q \text{ for } q \in Z$$

Then, when $n = k + 1$,

$$f(k+1) = f(n) = 3^{2n} - 1 = 3^{2(k+1)} - 1 = 3^{2k} \cdot 9 - 1 = (3^{2k} - 1 + 1) \cdot 9 - 1 = 9 \cdot (3^{2k} - 1) + 8$$
$$= 9 \cdot f(k) + 8 = 9 \cdot 8 \cdot q + 8 = 8 \cdot (9q + 1)$$

Let $t = 9q + 1$, we have $t \in Z$ and $f(k + 1) = 8 \cdot t$. Then, based on the definition of $a|b$, we have

$$8|f(k + 1)$$

Therefore, we prove, for each integer $n \geq 0$, $3^{2n} - 1$ is divisible by 8.

(b) For each integer $n \geq 2$, $2^n < (n + 1)!$.
✓
Proof by Mathematical Induction.
Let $f(n) = 2^n$ and $g(n) = (n + 1)!$. Then, we prove $f(n) < g(n)$ for $n \geq 2$.
Base Case: When $n = 2$,

$$f(2) = 2^n = 2^2 = 4; \; g(n) = (2 + 1)! = 3! = 3 * 2 * 1 = 6.$$

Therefore, $f(2) < g(2)$.
Induction Step.
Assume when $n = k$, we have

$$f(k) < g(k) \qquad i.e., \quad 2^k < (k + 1)!$$

Then, when $n = k + 1$,

$$f(k + 1) = f(n) = 2^n = 2^{k+1} = 2 * 2^k = 2 * f(k).$$

$$g(k + 1) = g(n) = (n + 1)! = (k + 1 + 1)! = (k + 2)! = (k + 2) * (k + 1)! = (k + 2) * g(k).$$

Because $k \geq 2$, we have $k * g(k) \geq 2 * 2^2 = 8 > 0$, and $f(k) - g(k) > 0$ from the assumption,

$$g(k+1) - f(k+1) = (k+2) * g(k) - 2 * f(k) = k * g(k) + 2 * (g(k) - f(k)) > 0 + 2 * 0 = 0$$

As a result, we have

$$f(k + 1) < g(k + 1).$$

Therefore, we prove, for each integer $n \geq 2$, $2^n < (n + 1)!$.

(c) For each integer $n \geq 0$, $1 + 3n \leq 4^n$.

✓

Proof by Mathematical Induction.

Let $f(n) = (1 + 3n)$ and $g(n) = 4^n$. Then, we prove $f(n) \leq g(n)$ for $n \geq 0$.

Base Case: When $n = 0$,

$$f(0) = (1 + 3 * 0) = 1; \ g(n) = 4^0 = 1.$$

Therefore, $f(0) = g(0)$.

Induction Step.

Assume when $n = k$, we have

$$f(k) \leq g(k) \qquad i.e., \quad 1 + 3k \leq 4^k$$

Then, when $n = k + 1$,

$$f(k+1) = f(n) = 1 + 3n = 1 + 3(k+1) = 3k + 1 + 3 = f(k) + 3$$

$$g(k+1) = g(n) = 4^n = 4^{k+1} = 4 * 4^k = 4 * g(k).$$

Because $k \geq 0$, $9 \cdot k \geq 0$, and $g(k) - f(k) \geq 0$ from the assumption, we have

$$g(k+1) - f(k+1) = 4 * g(k) - (f(k) + 3) = (g(k) - f(k)) + 3(f(k) - 1) = (g(k) - f(k)) + 9 \cdot k \geq 0 + 0 = 0$$

As a result, we have

$$f(k+1) \leq g(k+1).$$

Therefore, we prove, for each integer $n \geq 0$, $1 + 3n \leq 4^n$.

(d) For every real number $x > -1$ and every integer $n \geq 2$, $1 + nx \leq (1 + x)^n$.

✓

Proof by Mathematical Induction.

Let $f(n) = 1 + nx$ and $g(n) = (1 + x)^n$. Then, we prove $f(n) \leq g(n)$ for $n \geq 2$, where $x > -1$.

Base Case: When $n = 2$,

$$f(2) = 1 + 2x; \ g(2) = (1 + x)^2 = 1 + 2x + x^2.$$

Then,

$$f(2) - g(2) = (1 + 2x) - (1 + 2x + x^2) = -x^2.$$

Since $-x^2 \leq 0$ for any $x > -1$, we have $f(2) - g(2) \leq 0$. That is, $f(2) \leq g(2)$.

Induction Step.

Assume when $n = k$, we have

$$f(k) \leq g(k) \qquad i.e., \quad 1 + kx \leq (1 + x)^k$$

Then, when $n = k + 1$,

$$f(k+1) = f(n) = 1 + nx = 1 + (k+1)x = 1 + kx + x = f(k) + x$$

$$g(k+1) = g(n) = (1+x)^n = (1+x)^{k+1} = (1+x)*(1+x)^k = (1+x)*g(k) = g(k) + x*g(k).$$

In the following, we consider three cases to prove $x \leq x * g(k)$.

- Case 1: when $x = 0$, we have $x = x * g(k)$.
- Case 2: when $-1 < x < 0$, we have $0 < 1 + x < 1$ and $0 < g(k) = (1+x)^k < 1$. Then, we have
$$x * g(k) > x \ \because x < 0.$$
For example, if $g(k) = \frac{1}{2}$, $x = -\frac{1}{3}$, then $x * g(k) = -\frac{1}{3} * \frac{1}{2} = -\frac{1}{6} > -\frac{1}{3} = x$.
- Case 3: when $x > 0$, we have $1 + x > 1$ and $g(k) = (1+x)^k > 1$. Then, we have
$$x * g(k) > x \ \because x > 0.$$

Thus, we have $x \leq x * g(k)$. In addition, we have $f(k) \leq g(k)$. We can deduce that $f(k) + x \leq g(k) + x * g(k)$, i.e., $f(k+1) \leq g(k+1)$.
Therefore, we prove, for each integer $n \geq 2$, $1 + nx \leq (1+x)^n$, where $x > -1$.

[**20**] 6. Prove each of the following statements using strong mathematical induction.

(a) Suppose $b_1, b_2, b_3, \cdots$ is a sequence defined as follows:

$$b_1 = 4, \quad b_2 = 12, \qquad b_n = b_{n-2} + b_{n-1} \qquad \text{for each integer } n \geq 3$$

Prove that b_n is divisible by 4 for every integer $n \geq 1$.
✓
Proof by Strong Mathematical Induction.
Let $a|b$ be b is divisible by a, i.e., $b = a \cdot k$ for some $k \in Z$.
Base Case: When $n = 1, 2$, because $b_1 = 4, b_2 = 12$, we have $b_1 = 4 \cdot k_1$, where $k_1 = 1$, $b_2 = 4 \cdot k_2$, where $k_2 = 3$. Based on the definition of $a|b$, we have

$$4|b_1, \quad 4|b_2.$$

Induction Step.
Assume when all $n \leq k$, we have

$$4|b_n$$

Then, when $n = k + 1$, from $b_n = b_{n-2} + b_{n-1}$ for each integer $n \geq 3$, we have

$$b_{k+1} = b_n = b_{n-2} + b_{n-1} = b_{k-1} + b_k$$

From the assumption, for all $n \leq k$, $4|b_n$, we have

$$4|b_{k-1}, \quad and \quad 4|b_k$$

That is, $b_{k-1} = 4 \cdot q_{k-1}$, where $q_{k-1} \in Z$, $b_k = 4 \cdot q_k$, where $q_k \in Z$. Then,

$$b_{k+1} = b_{k-1} + b_k = 4 \cdot q_{k-1} + 4 \cdot q_k = 4 \cdot (q_{k-1} + q_k)$$

Let $q_{k+1} = q_{k-1} + q_k$, it is easy to see $q_{k+1} \in Z$. Therefore, based on the definition of $a|b$, we have

$$4|b_{k+1}$$

The proof is completed.

(b) Suppose $f_0, f_1, f_2, \cdots$ is a sequence defined as follows:

$$f_0 = 5, \quad f_1 = 16, \qquad f_n = 7f_{n-1} - 10f_{n-2} \qquad \text{for each integer } n \geq 2$$

Prove that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for every integer $n \geq 0$.

✓

Proof by Strong Mathematical Induction.

Base Case: When $n = 0, 1$, we have

$$f_0 = 5 = 3 \cdot 2^0 + 2 \cdot 5^0,$$
$$f_1 = 16 = 3 \cdot 2^1 + 2 \cdot 5^1.$$

Induction Step. Assume when all $n \leq k$, we have

$$f_n = 3 \cdot 2^n + 2 \cdot 5^n.$$

Then, when $n = k + 1$, from $f_{k+1} = f_n = 7f_{n-1} - 10f_{n-2}$ for each integer $n \geq 3$, we have

$$f_{k+1} = 7f_{n-1} - 10f_{n-2} = 7f_k - 10f_{k-1}.$$

From the assumption, for all $n \leq k$, $f_n = 3 \cdot 2^n + 2 \cdot 5^n$, we have

$$f_k = 3 \cdot 2^k + 2 \cdot 5^k \quad and \quad f_{k-1} = 3 \cdot 2^{k-1} + 2 \cdot 5^{k-1}.$$

As a result,

$$\begin{aligned} f_{k+1} &= 7f_k - 10f_{k-1} \\ &= 7 \cdot (3 \cdot 2^k + 2 \cdot 5^k) - 10 \cdot (3 \cdot 2^{k-1} + 2 \cdot 5^{k-1}) \\ &= 7 \cdot 3 \cdot 2^k + 7 \cdot 2 \cdot 5^k - 10 \cdot 3 \cdot 2^{k-1} - 10 \cdot 2 \cdot 5^{k-1} \\ &= (7 \cdot 3 - \frac{10 \cdot 3}{2})2^k + (7 \cdot 2 - \frac{10 \cdot 2}{5}) \cdot 5^k \\ &= 3 \cdot 2 \cdot 2^k + 2 \cdot 5 \cdot 5^k \\ &= 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}. \end{aligned}$$

The proof is completed.