University of New Brunswick
Faculty of Computer Science
CS1303: Discrete Structures
Homework Assignment 6, ***Due Time, Date*** *11:59 PM, April 2, 2021*

Student Name: ____________________________ Matriculation Number: ________________

Instructor: Rongxing Lu
The marking scheme is shown in the left margin and [100] constitutes full marks.

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[16] 1. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:

(a) $A \cup B$
(b) $A \cap B$
(c) $A \cup C$
(d) $A \cap C$
(e) $A - B$
(f) $B - A$
(g) $B \cup C$
(h) $B \cap C$

[8] 2. Let S be the set of all strings of 0's and 1's of length 4, and let A and B be the following subsets of S: $A = \{1110, 1111, 1000, 1001\}$ and $B = \{1100, 0100, 1111, 0111\}$. Find each of the following:

(a) $A \cup B$
(b) $A \cap B$
(c) $A - B$
(d) $B - A$

[20] 3. In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions.

(a) $A \subseteq B$, $C \subseteq B$, $A \cap C = \varnothing$
(b) $B \subseteq A$, $B \cap C = \varnothing$
(c) $A \cap B = \varnothing$, $A \subseteq C$, $C \cap B \neq \varnothing$
(d) $A \cap B \neq \varnothing$, $B \cap C \neq \varnothing$, $A \cap C = \varnothing$, $A \not\subseteq B$, $C \not\subseteq B$

[16] 4. Let $A = \{a, b\}$, $B = \{1, 2\}$, $C = \{2, 3\}$. Find each of the following sets.

(a) $A \times (B \cup C)$
(b) $(A \times B) \cup (A \times C)$
(c) $A \times (B \cap C)$

(d) $(A \times B) \cap (A \times C)$

[**10**] 5. Let Z be the set of all integers and let

$$A_0 = \{n \in Z | n = 4k + 0, k \in Z\}$$

$$A_1 = \{n \in Z | n = 4k + 1, k \in Z\}$$

$$A_2 = \{n \in Z | n = 4k + 2, k \in Z\}$$

$$A_3 = \{n \in Z | n = 4k + 3, k \in Z\}$$

Is (A_0, A_1, A_2, A_3) a partition of Z? Explain your answer.

[**20**] 6. Assume that all sets are subsets of a universal set U. Please prove each statement below.

(a) For all sets A, B, and C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \varnothing$.

(b) For all sets A, B, C, and D, if $A \cap C = \varnothing$, then $(A \times B) \cap (C \times D) = \varnothing$.

(c) For every positive integer n, if A and $B_1, B_2, B_3, \cdots$ are any sets, then

$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

(d) For every positive integer n, if A and $B_1, B_2, B_3, \cdots$ are any sets, then

$$\bigcup_{i=1}^{n} (A \times B_i) = A \times \left(\bigcup_{i=1}^{n} B_i \right)$$

[**10**] 7. Find a counterexample to show that the each statement is false. Assume all sets are subsets of a universal set U.

(a) For all sets A, B, and C,

$$(A \cup B) \cap C = A \cup (B \cap C).$$

(b) For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$ then $A \not\subseteq C$.

Solutions.

[**16**] 1. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:

(a) $A \cup B$

✓

$A \cup B = \{1, 3, 5, 6, 7, 9\}$

(b) $A \cap B$

✓

$A \cap B = \{3, 9\}$

(c) $A \cup C$

✓

$A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(d) $A \cap C$

✓

$A \cap C = \varnothing$

(e) $A - B$

✓

$A - B = \{1, 5, 7\}$

(f) $B - A$

✓

$B - A = \{6\}$

(g) $B \cup C$

✓

$B \cup C = \{2, 3, 4, 6, 8, 9\}$

(h) $B \cap C$

✓

$B \cap C = \{6\}$

[**8**] 2. Let S be the set of all strings of 0's and 1's of length 4, and let A and B be the following subsets of S: $A = \{1110, 1111, 1000, 1001\}$ and $B = \{1100, 0100, 1111, 0111\}$. Find each of the following:

(a) $A \cup B$

✓

$A \cup B = \{1110, 1111, 1000, 1001, 1100, 0100, 0111\}$

(b) $A \cap B$

✓

$A \cap B = \{1111\}$

(c) $A - B$

✓

$A - B = \{1110, 1000, 1001\}$

(d) $B - A$

✓

$B - A = \{1100, 0100, 0111\}$

[**20**] 3. In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions.

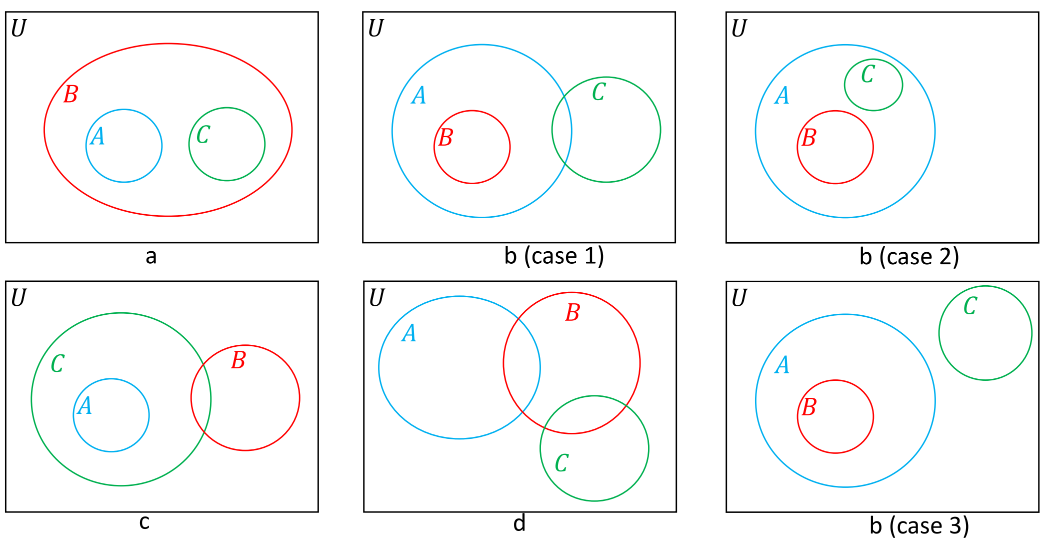
(a) $A \subseteq B, C \subseteq B, A \cap C = \varnothing$

(b) $B \subseteq A, B \cap C = \varnothing$

(c) $A \cap B = \varnothing, A \subseteq C, C \cap B \neq \varnothing$

(d) $A \cap B \neq \varnothing, B \cap C \neq \varnothing, A \cap C = \varnothing, A \nsubseteq B, C \nsubseteq B$

✓



[**16**] 4. Let $A = \{a, b\}, B = \{1, 2\}, C = \{2, 3\}$. Find each of the following sets.

(a) $A \times (B \cup C)$

✓

$A \times (B \cup C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

(b) $(A \times B) \cup (A \times C)$

✓

$(A \times B) \cup (A \times C) = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

(c) $A \times (B \cap C)$

✓

$A \times (B \cap C) = \{(a, 2), (b, 2)\}$

(d) $(A \times B) \cap (A \times C)$

✓

$(A \times B) \cap (A \times C) = \{(a, 2), (b, 2)\}$

[**10**] 5. Let Z be the set of all integers and let

$$A_0 = \{n \in Z | n = 4k + 0, k \in Z\}$$

$$A_1 = \{n \in Z | n = 4k + 1, k \in Z\}$$

$$A_2 = \{n \in Z | n = 4k + 2, k \in Z\}$$

$$A_3 = \{n \in Z | n = 4k + 3, k \in Z\}$$

Is (A_0, A_1, A_2, A_3) a partition of Z? Explain your answer.

✓

Yes, (A_0, A_1, A_2, A_3) is a partition of Z.

By the quotient-remainder theorem, every integer $n \in Z$ can be represented in exactly one of the four forms

$$n = 4k \quad or \quad n = 4k + 1 \quad or \quad n = 4k + 2 \quad or \quad n = 4k + 3,$$

for some integer k. This implies that no integer can be in any two or more of the sets A_0, A_1, A_2, or A_3. So, A_0, A_1, A_2, and A_3 are mutually disjoint. The theorem also implies that every integer must be in one of the sets A_0, A_1, A_2, or A_3. So $Z = A_0 \cup A_1 \cup A_2 \cup A_3$.

[**20**] 6. Assume that all sets are subsets of a universal set U. Please prove each statement below.

(a) For all sets A, B, and C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \varnothing$.

✓

P: For all sets A, B, and C,

$$(B \cap C \subseteq A) \rightarrow ((C - A) \cap (B - A) = \varnothing)$$

Negation:

$\neg P$: There exist sets A, B, and C,

$$(B \cap C \subseteq A) \wedge ((C - A) \cap (B - A) \neq \varnothing)$$

Proof by Contradiction.

Suppose $\neg P$ is true. Then, there exist sets A, B, and C, such that $((C - A) \cap (B - A) \neq \varnothing$. That is,

$$\exists x, x \in (C - A) \cap (B - A)$$

$$\equiv \exists x, (x \in (C - A)) \wedge (x \in (B - A))$$
$$\equiv \exists x, (x \in C \wedge x \in A^c) \wedge (x \in B \wedge x \in A^c)$$
$$\equiv \exists x, (x \in B) \wedge (x \in C) \wedge (x \in A^c)$$
$$\equiv \exists x, (x \in B \cap C) \wedge (x \in A^c)$$

Because $\exists x, x \in B \cap C$, and also $B \cap C \subseteq A$, we have $x \in A$. However, as $x \in A^c$, we have $(x \in A) \wedge (x \in A^c) = \mathbf{c}$ draws a contradiction. Therefore, $\neg P$ is false, and P is true. That is,
For all sets A, B, and C,

$$(B \cap C \subseteq A) \rightarrow ((C - A) \cap (B - A) = \varnothing)$$

(b) For all sets A, B, C, and D, if $A \cap C = \varnothing$, then $(A \times B) \cap (C \times D) = \varnothing$.
✓
P: For all sets A, B, C, and D,

$$(A \cap C = \varnothing) \rightarrow ((A \times B) \cap (C \times D) = \varnothing)$$

Negation:
$\neg P$: There exist sets A, B, C, and D,

$$(A \cap C = \varnothing) \wedge ((A \times B) \cap (C \times D) \neq \varnothing)$$

Proof by Contradiction.
Suppose $\neg P$ is true. Then, there exist sets A, B, C, and D such that $(A \times B) \cap (C \times D) \neq \varnothing$. That is,

$$\exists (x, y), (x, y) \in (A \times B) \cap (C \times D)$$
$$\equiv \exists (x, y), ((x, y) \in (A \times B)) \wedge ((x, y) \in (C \times D))$$
$$\equiv \exists (x, y), (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D)$$
$$\equiv \exists (x, y), (x \in A \cap C) \wedge (y \in B \cap D)$$

Because $\exists x, x \in A \cap C$, we have $A \cap C \neq \varnothing$. Then, $(A \cap C \neq \varnothing) \wedge (A \cap C = \varnothing) = \mathbf{c}$ draws a contradiction. Therefore, $\neg P$ is false, and P is true. That is,
For all sets A, B, C, and D,

$$(A \cap C = \varnothing) \rightarrow ((A \times B) \cap (C \times D) = \varnothing)$$

(c) For every positive integer n, if A and $B_1, B_2, B_3, \cdots$ are any sets, then

$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

✓
We prove $A \cap (\bigcup_{i=1}^{n} B_i) = \bigcup_{i=1}^{n} (A \cap B_i)$ by respectively proving $A \cap (\bigcup_{i=1}^{n} B_i) \subseteq \bigcup_{i=1}^{n} (A \cap B_i)$ and $\bigcup_{i=1}^{n} (A \cap B_i) \subseteq A \cap (\bigcup_{i=1}^{n} B_i)$ as follows.

(1) Prove $A \cap (\bigcup_{i=1}^{n} B_i) \subseteq \bigcup_{i=1}^{n}(A \cap B_i)$
If $A \cap (\bigcup_{i=1}^{n} B_i) = \varnothing$, the result is straightforward. So, we consider $A \cap (\bigcup_{i=1}^{n} B_i) \neq \varnothing$.
Because $A \cap (\bigcup_{i=1}^{n} B_i) \neq \varnothing$,

$$\begin{aligned}
&\exists x, x \in A \cap \left(\bigcup_{i=1}^{n} B_i\right) \\
&\equiv \exists x, (x \in A) \wedge (x \in \left(\bigcup_{i=1}^{n} B_i\right)) \\
&\equiv \exists x, (x \in A) \wedge (x \in B_1 \vee x \in B_2 \vee \cdots \vee x \in B_n) \\
&\equiv \exists x, (x \in A \wedge x \in B_1) \vee (x \in A \wedge x \in B_2) \vee \cdots \vee (x \in A \wedge x \in B_n) \\
&\equiv \exists x, (x \in A \cap B_1) \vee (x \in A \cap B_2) \vee \cdots \vee (x \in A \cap B_n) \\
&\equiv \exists x, x \in \bigcup_{i=1}^{n}(A \cap B_i)
\end{aligned}$$

(2) Prove $\bigcup_{i=1}^{n}(A \cap B_i) \subseteq A \cap (\bigcup_{i=1}^{n} B_i)$
If $\bigcup_{i=1}^{n}(A \cap B_i) = \varnothing$, the result is straightforward. So, we consider $\bigcup_{i=1}^{n}(A \cap B_i) \neq \varnothing$.
Because $\bigcup_{i=1}^{n}(A \cap B_i) \neq \varnothing$,

$$\begin{aligned}
&\exists x, x \in \bigcup_{i=1}^{n}(A \cap B_i) \\
&\equiv \exists x, (x \in A \cap B_1) \vee (x \in A \cap B_2) \vee \cdots \vee (x \in A \cap B_n) \\
&\equiv \exists x, (x \in A \wedge x \in B_1) \vee (x \in A \wedge x \in B_2) \vee \cdots \vee (x \in A \wedge x \in B_n) \\
&\equiv \exists x, (x \in A) \wedge (x \in B_1 \vee x \in B_2 \vee \cdots \vee x \in B_n) \\
&\equiv \exists x, (x \in A) \wedge (x \in \left(\bigcup_{i=1}^{n} B_i\right)) \\
&\equiv \exists x, x \in A \cap \left(\bigcup_{i=1}^{n} B_i\right)
\end{aligned}$$

The proof is completed.

(d) For every positive integer n, if A and $B_1, B_2, B_3, \cdots$ are any sets, then

$$\bigcup_{i=1}^{n}(A \times B_i) = A \times \left(\bigcup_{i=1}^{n} B_i\right)$$

✓
We prove $\bigcup_{i=1}^{n}(A \times B_i) = A \times (\bigcup_{i=1}^{n} B_i)$ by respectively proving $\bigcup_{i=1}^{n}(A \times B_i) \subseteq A \times (\bigcup_{i=1}^{n} B_i)$ and $A \times (\bigcup_{i=1}^{n} B_i) \subseteq \bigcup_{i=1}^{n}(A \times B_i)$ as follows.

(1) Prove $\bigcup_{i=1}^{n}(A \times B_i) \subseteq A \times (\bigcup_{i=1}^{n} B_i)$
If $\bigcup_{i=1}^{n}(A \times B_i) = \varnothing$, the result is straightforward. So, we consider $\bigcup_{i=1}^{n}(A \times B_i) \neq \varnothing$.
Because $\bigcup_{i=1}^{n}(A \times B_i) \neq \varnothing$,

$$\begin{aligned} & \exists(x,y),(x,y) \in \bigcup_{i=1}^{n}(A \times B_i) \\ \equiv\ & \exists(x,y),((x,y) \in (A \times B_1)) \vee ((x,y) \in (A \times B_2)) \vee \cdots \vee ((x,y) \in (A \times B_n)) \\ \equiv\ & \exists(x,y),(x \in A \wedge y \in B_1) \vee (x \in A \wedge y \in B_2) \vee \cdots \vee (x \in A \wedge y \in B_n) \\ \equiv\ & \exists(x,y),(x \in A) \wedge (y \in B_1 \vee y \in B_2 \vee \cdots \vee y \in B_n) \\ \equiv\ & \exists(x,y),(x \in A) \wedge (y \in \left(\bigcup_{i=1}^{n} B_i\right)) \\ \equiv\ & \exists(x,y),(x,y) \in A \times \left(\bigcup_{i=1}^{n} B_i\right) \end{aligned}$$

(2) Prove $A \times (\bigcup_{i=1}^{n} B_i) \subseteq \bigcup_{i=1}^{n}(A \times B_i)$
If $A \times (\bigcup_{i=1}^{n} B_i) = \varnothing$, the result is straightforward. So, we consider $A \times (\bigcup_{i=1}^{n} B_i) \neq \varnothing$.
Because $A \times (\bigcup_{i=1}^{n} B_i) \neq \varnothing$,

$$\begin{aligned} & \exists(x,y),(x,y) \in A \times \left(\bigcup_{i=1}^{n} B_i\right) \\ \equiv\ & \exists(x,y),(x \in A) \wedge (y \in \left(\bigcup_{i=1}^{n} B_i\right)) \\ \equiv\ & \exists(x,y),(x \in A) \wedge (y \in B_1 \vee y \in B_2 \vee \cdots \vee y \in B_n) \\ \equiv\ & \exists(x,y),(x \in A \wedge y \in B_1) \vee (x \in A \wedge y \in B_2) \vee \cdots \vee (x \in A \wedge y \in B_n) \\ & \exists(x,y),(x,y) \in \bigcup_{i=1}^{n}(A \times B_i) \end{aligned}$$

The proof is completed.

[**10**] 7. Find a counterexample to show that the each statement is false. Assume all sets are subsets of a universal set U.

(a) For all sets A, B, and C,

$$(A \cup B) \cap C = A \cup (B \cap C).$$

✓

Negation:
There exist sets A, B, and C,

$$(A \cup B) \cap C \neq A \cup (B \cap C).$$

Counterexample: Let $A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$, and $C = \{1, 2, 5\}$. Then, we have

$$(A \cup B) \cap C = \{1, 2, 3, 4\} \cap \{1, 2, 5\} = \{1, 2\}.$$
$$A \cup (B \cap C) = \{1, 2, 3\} \cup \{1, 2\} = \{1, 2, 3\}.$$

Hence, $(A \cup B) \cap C \neq A \cup (B \cap C)$.

(b) For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$ then $A \not\subseteq C$.

✓

For all sets A, B, and C,

$$(A \not\subseteq B) \wedge (B \not\subseteq C) \rightarrow (A \not\subseteq C)$$

Negation:

There exist sets A, B, and C,

$$(A \not\subseteq B) \wedge (B \not\subseteq C) \wedge (A \subseteq C)$$

Counterexample: Let $A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$, and $C = \{1, 2, 3, 5, 6\}$. Then, we have

$$A \not\subseteq B \quad \text{and} \quad B \not\subseteq C, \quad \text{but} \quad A \subseteq C.$$