Chapter 3

Predicate Logic

> *"Logic will get you from A to B. Imagination will take you everywhere." A. Einstein*

In the previous chapter, we studied propositional logic. This chapter is dedicated to another type of logic, called *predicate logic*.

Let us start with a motivating example.

Example 21. Consider the following two statements:

- Every SCE student must study discrete mathematics.
- Jackson is an SCE student.

It looks "logical" to deduce that therefore, Jackson must study discrete mathematics. However, this cannot be expressed by propositional logic...you may try it, but you can already notice that none of the logical operators we have learnt are applicable here.

We need new tools!

Definition 20. A predicate is a statement that contains variables (predicate variables), and they may be true or false depending on the values of these variables.

Example 22. $P(x) =$ "x^2 is greater than x" is a predicate. It contains one predicate variable x. If we choose $x = 1$, $P(1)$ is "1 is greater than 1", which is a proposition (always false).

Limitation Of Propositional Logic

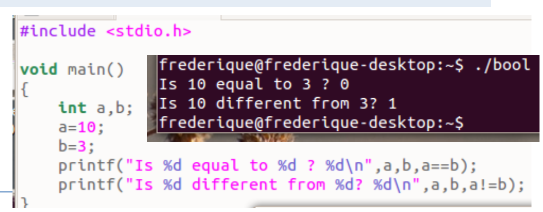
- Every SCE student must study discrete mathematics
- Jackson is a SCE student
  - So Jackson must study discrete mathematics
    - This idea *can't be expressed* with propositional logic
- What propositional logic allows to express:
  - If Jackson is a SCE student he must study discrete mathematics
  - Jackson is a SCE student
    - So Jackson must study discrete mathematics

Predicates

- Is the statement "x^2 is greater than x" a proposition?
- Define P(x) = '*x^2 is greater than x*'.
- Is P(1) a proposition? P(1)="1^2 is greater than 1" (F)

A predicate is a statement that contains variables (predicate variables) and that may be true or false depending on the values of these variables.

- P(x) is a predicate.



Since a predicate takes value true or false once instantiated (that is, once its variables are taking values), we may alternatively say that a predicate instantiated becomes a proposition.

It is needed to explicit which are the values that a predicate variable can possibly take.

Definition 21. The domain of a predicate variable is the collection of all possible values that the variable may take.

Example 23. Consider the predicate $P(x) =$ "x^2 is greater than x". Then the domain of x could be for example the set $\mathbb{Z}$ of all integers. It could alternatively be the set $\mathbb{R}$ of real numbers. Whether instantiations of a predicate are true or false may depend on the domain considered.

When several predicate variables are involved, they may or not have different domains.

Example 24. Consider the predicate $P(x, y) =$ "$x > y$", in two predicate variables. We have $\mathbb{Z}$ (the set of integers) as domain for both of them.

- Take $x = 4$, $y = 3$, then $P(4, 3) =$ "$4 > 3$", which is a proposition taking the value true.
- Take $x = 1$, $y = 2$, then $P(1, 2) =$ "$1 > 2$", which is a proposition taking the value false.
- Note that in general $P(x, y) \neq P(y, x)$!

We now introduce two quantifiers (describing "parts or quantities" from a domain), the universal quantification and the existential quantification.

Definition 22. A universal quantification is a quantifier meaning "given any" or "for all". We use the following symbol:

$$\boxed{\forall \text{ (universal quantification)}}$$

Example 25. Here is a formal way to say that for all values that a predicate variable x can take in a domain D, the predicate is true:

$$\underbrace{\forall x}_{\text{for all } x} \underbrace{\in D}_{\text{belonging to } D}, \ P(x) \text{ (is true)}$$

For example

$$\underbrace{\forall x}_{\text{for all } x} \underbrace{\in \mathbb{R}}_{\text{belonging to the real numbers}}, \ x^2 \geq 0.$$

Predicate Instantiated/Domain

A predicate is a statement that contains variables (predicate variables) and that may be true or false depending on the values of these variables.

- A predicate instantiated (where variables are evaluated in specific values) is a proposition.

The domain of a predicate variable is the collection of all possible values that the variable may take.

  - e.g. the domain of x in P(x): integer
  - Different variables may have different domains.

- Predicate logic extends (is more powerful than) propositional logic.

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Example

- Let P(x, y) = "x > y".
  Domain: integers, i.e. both x and y are integers.
- P(4, 3) means "4 >3", so P(4, 3) is TRUE;
- P(1, 2) means "1>2" , so P(1, 2) is FALSE;
- P(3,4) is false (in general, P(x,y) and P(y,x) not equal).

Quantification

- Statements like
  - Some birds are angry.
  - On the internet, no one knows who you are.
  - The square of any real number is nonnegative.

Universal Quantification

A universal quantification is a quantifier (something that tells the amount or quantity) meaning "given any" or "for all".

Symbol: ∀

- E.g. "$\forall x \in D$ P(x) is true" *iff* "*P*(*x*) is true for every *x* in *D*".
  - ∀ universal quantifiers, "for all", "for every"
  - ∈ - "is a member (or) element of", "belonging to"
  - **D** – domain of predicate variable
- The square of any real number is nonnegative.

$\forall x \in R,\ x^2 \geq 0.$

Existential Quantification

An existential quantification is a quantifier (something that tells the amount or quantity) meaning “there exists“, “there is at least one“ or “for some”.

Symbol: ∃

- E.g. “∃ x∈D, P(x) is true” *iff* “*P*(*x*) is true for *at least one x* in *D”.*
    - ∃ existential quantifier, “there exists”
- Some birds are angry.
    - D={birds}, P(x)=”x is angry”.

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Nested Quantification (I)

- A proposition may contain multiple quantifiers
    - “All rabbits are faster than all tortoises.”
    - Domains: R={rabbits}, T={tortoises}
    - Predicate C(x, y): Rabbit x is faster than tortoise y
- In symbols
    - **∀x∈R (∀y∈T, C(x, y))** or **∀x∈R, ∀y∈T, C(x, y)**
- In words
    - For any rabbit *x,* and for any tortoise *y*, *x* is faster than *y.*

Definition 23. An existential quantification is a quantifier meaning "there exists", "there is at least one" or "for some". We use the following symbol:

$$\boxed{\exists \text{ (existential quantification)}}$$

Example 26. Here is a formal way to say that for some values that a predicate variable x can take in a domain D, the predicate is true:

$$\underbrace{\exists x}_{\text{for some } x} \underbrace{\in D}_{\text{belonging to } D},\ P(x) \text{ (is true)}$$

For example, for $D = \{$ birds $\}$, $P(x) =$ "x is angry",

$$\underbrace{\exists x}_{\text{Some}} \underbrace{\in D}_{\text{birds}},\ \underbrace{P(x) \text{ (is true)}}_{\text{are angry}}.$$

The term nested quantification refers to statements involving several quantifiers. Here is a series of examples.

Example 27. All statements involve two predicate variables x and y, where x has for domain $R = \{$ rabbits $\}$, while y has for domain $T = \{$ tortoises $\}$. The predicate used is $C(x, y) =$"Rabbit x is faster than tortoise y".

- In logic symbolism, we write "All rabbits are faster than all tortoises":

$$\underbrace{\forall x}_{\text{For any}} \underbrace{\in R}_{\text{rabbit } x},\ \underbrace{\forall y}_{\text{for any}} \underbrace{\in T}_{\text{tortoise } y}\ \underbrace{C(x, y) \text{ (is true)}}_{x \text{ is faster than } y}.$$

- In logic symbolism, we write "Every rabbit is faster than some tortoise":

$$\underbrace{\forall x}_{\text{For any}} \underbrace{\in R}_{\text{rabbit } x},\ \underbrace{\exists y}_{\text{there is}} \underbrace{\in T}_{\text{a tortoise } y}\ \underbrace{C(x, y) \text{ (is true)}}_{\text{such that } x \text{ is faster than } y}.$$

- In logic formalism, we write "There is a rabbit which is faster than all tortoises":

$$\underbrace{\exists x}_{\text{There exists}} \underbrace{\in R}_{\text{a rabbit } x \text{ such that}},\ \underbrace{\forall y}_{\text{for any}} \underbrace{\in T}_{\text{tortoise } y}\ \underbrace{C(x, y) \text{ (is true)}}_{x \text{ is faster than } y}.$$

Nested Quantification (II)

- Another example
  - "Every rabbit is faster than some tortoise."
  - Domains: R={rabbits}, T={tortoises}.
  - Predicate C(x, y): Rabbit x is faster than tortoise y
- In symbols
  - $\forall x \in R\ (\exists y \in T,\ C(x, y))$ **or** $\forall x \in R,\ \exists y \in T,\ C(x, y)$
- In words:
  - For any rabbit x, there exists a (some) tortoise y, such that x is faster than y.

Nested Quantification (III)

- Another example
  - "There is a rabbit which is faster than all tortoises."
  - Domains: R={rabbits}, T={ tortoises}.
  - Predicate C(x, y): Rabbit x is faster than tortoise y.
- In symbols (note the ordering in nesting)
  - $\exists\, x \in R\ (\forall y \in T,\ C(x, y))$
- In words:
  - There exists a rabbit x, such that for any tortoise y, this rabbit x is faster than y.

The same way we assigned truth values to propositions, we may assign truth values to quantified statements.

- $(\forall\ x \in D,\ P(x))$ is true exactly when $P(x)$ is true for every $x \in D$. Thus it is false whenever there is at least one x for which $P(x)$ is false. Formally, for $D = \{x_1, \ldots, x_n\}$, we have the following equivalence:
$$(\forall\ x \in D,\ P(x)) \equiv (P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)).$$
- $(\exists\ x \in D,\ P(x))$ is true exactly when $P(x)$ is true for at least one $x \in D$. Thus it is false when $P(x)$ is false for all $x \in D$. Formally, for $D = \{x_1, \ldots, x_n\}$, we have the following equivalence:
$$(\exists\ x \in D,\ P(x)) \equiv (P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)).$$

Since $(\exists\ x \in D,\ P(x))$ takes truth values, it can also be negated, that is

$$\boxed{\neg(\exists\ x \in D,\ P(x)) \equiv \forall x \in D,\ \neg P(x),}$$

where the equivalence follows from the fact that $(\exists\ x \in D,\ P(x))$ is false whenever for all $x \in D$ $P(x)$ is false. Similarly

$$\boxed{\neg(\forall\ x \in D,\ P(x)) \equiv \exists\ x \in D,\ \neg P(x)} \tag{3.1}$$

since $(\forall\ x \in D,\ P(x))$ is false whenever there is one $x \in D$ for which $P(x)$ is false. See Exercise 23 for negating a statement involving several quantifiers.

We can similarly assign truth values to combinations of predicates, or negation of combinations of predicates. The equivalence

$$\neg(\forall\ x \in D,\ P(x) \wedge Q(x)) \equiv \exists\ x \in D,\ \neg(P(x) \wedge Q(x))$$

holds, setting $P'(x) = P(x) \wedge Q(x)$ and using (3.1) on $P'(x)$. Now $(P(x) \wedge Q(x))$ is a proposition for any instantiation of x, thus we can apply De Morgan laws:

$$\neg(\forall\ x \in D,\ P(x) \wedge Q(x)) \equiv \exists\ x \in D,\ \neg P(x) \vee \neg Q(x).$$

Suppose now you are given a statement involving quantifies, whose truth table has to be determined. There are several ways to do so: (1) Method of Exhaustion, (2) Method of Case, and (3) Method of Logic Derivation.

Method of exhaustion: if the domain contains a small number of elements, try them all! For example, if $D = \{5, 6, 7, 8, 9\}$, and $P(x) =$ "$x \in D,\ x^2 = x$", then just compute x^2 for all the values of $x \in D$ to conclude that this false.

Truth Value of Quantified Statements

Statement	When true	When false
$\forall x \in D, P(x)$	P(x) is true for every x.	There is one x for which P(x) is false.
$\exists x \in D, P(x)$	There is one x for which P(x) is true.	P(x) is false for every x.

Assume that D consists of $x_1, x_2, \ldots, x_n$

- $\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
- $\exists x \in D, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Negation of Quantification

- 'Not all SCE students study hard' = 'There is at least one SCE student who does not study hard'

$$\neg (\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$

- Negation of a universal quantification becomes an existential quantification.
- 'It is not the case that some students in this class are from NUS.'= 'All students in this class are not from NUS'

$$\neg (\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

- Negation of an existential quantification becomes an universal quantification.

Negation of Quantification

$\neg (\forall x \in D, P(x) \wedge Q(x))$

$\equiv \quad \exists x \in D\ \neg (P(x) \wedge Q(x))$ (negation of quantification)

$\equiv \quad \exists x \in D\ (\neg P(x) \vee \neg Q(x))$ (DeMorgan)

- Example: Not all students in this class are using Facebook and (also) Google+
  - There is some (at least one) student in this class who is not using Facebook or not using Google+ (or may be using neither)

How To Determine Truth Value

Statement	When true	When false
$\forall x \in D, P(x)$	P(x) is true for every x.	There is one x for which P(x) is false.
$\exists x \in D, P(x)$	There is one x for which P(x) is true.	P(x) is false for every x.

- Systematic approaches:
  – Method of exhaustion
  – Method of case
  – Method of logic derivation

Method of Exhaustion

- Let D={5,6,7,8,9}. Is $\exists\, x \in D, x^2=x$ true or false?
  - $5^2=25\neq 5$, $6^2=36\neq 6$, $7^2=49\neq 7$, $8^2=64\neq 8$, $9^2=81\neq 9$
  - So, false!
- Limitation?
  - Domain may be too large to try out all options
    - E.g., all integers

Method of Case

- Positive examples to prove existential quantification
- Let Z denote all integers. Is $\exists\, x \in Z, x^2=x$ *true or false?*
  - Take x =0 or 1 and we have it. True.

- Counterexample to disprove universal quantification
- Let R denote all reals. Is $\forall x \in R, x^2 > x$ *true or false?*
  - Take x=0.3 as a counterexample. False.

Positive example is **not** a proof of universal quantification

Negative example **is not** disproof of existential quantification

May be **hard** to find suitable "cases" even if such cases do exist!

Method of Case. Suppose you want to show that the truth value of $(\exists\ x, P(x))$ is true. For this, you just need to find one case, one instantiation of x, for which $P(x)$ is true.

Example 28. $P(x) =$ "$\exists\ x \in \mathbb{Z},\ x^2 = x$ is true, take $x = 1$ for example. Thus such an x exists.

Similarly, if you want to show that the truth value of $(\forall\ x, P(x))$ is false, it is enough to find one counterexmaple.

Example 29. $P(x) =$ "$\forall\ x \in \mathbb{R},\ x^2 > x$ is false, take $x = 0.3$ for example. Thus $P(x)$ cannot be true for all x.

However, you **cannot** show that $(\exists\ x, P(x))$ is false using some examples, you need to prove that you cannot find a single x in your domain for which $P(x)$ is true! Vice-versa, you cannot show that $(\forall\ x, P(x))$ is true by giving some examples, you need to show that this is always true, for every x in the domain considered.

To do this, again, if the domain is small, one may use the exhaustion method of trying all options, but if the domain is big (or infinite), like $\mathbb{Z}$, we need another method.

Method of Logic Derivation. This method consists of using logical steps to transform one logical expression into another.

Example 30. Suppose you want to know the truth value of $\exists\ x,\ (P(x) \vee Q(x))$, x has for domain $D = \{x_1, \ldots, x_n\}$. Then $\exists\ x,\ (P(x) \vee Q(x))$ is true if there is an $x_i \in D$ for which $P(x_i) \vee Q(x_i)$ is true that is

$$(\exists\ x,\ (P(x) \vee Q(x))) \equiv (P(x_1) \vee Q(x_1)) \vee \ldots \vee (P(x_n) \vee Q(x_n))$$

but now this new expression becomes true exactly when at least one $P(x_i)$ or $Q(x_j)$ is true, that is

$$(\exists\ x,\ (P(x) \vee Q(x))) \equiv (\exists\ x,\ P(x)) \vee (\exists\ x,\ Q(x)).$$

When trying to derive logic steps involving quantifiers, one should be **very careful** with the ordering of the quantifiers...A typical example is that

$$\forall\ x,\ \exists\ y,\ P(x,y) \equiv \exists\ y,\ \forall\ x,\ P(x,y)$$

does **not hold** in general!

Example 31. Consider the predicate $P(x,y) =$ "x admires y". Then $\forall\ x,\ \exists\ y,\ P(x,y)$ means that everyone admires someone, while $\exists\ y,\ \forall\ x,\ P(x,y)$ means that there exists one person who is admired by everyone!

Method Of Logic Derivation

Consider an (arbitrary) domain with n members. Is $\exists x(P(x) \vee Q(x))$ logically equivalent to $\exists xP(x) \vee \exists xQ(x)$?

$$\exists x(P(x) \vee Q(x))$$

$$\equiv [P(x_1) \vee Q(x_1)] \vee \ldots \vee [P(x_n) \vee Q(x_n)]$$

$$\equiv [P(x_1) \vee \ldots \vee P(x_n)] \vee [Q(x_1) \vee \ldots \vee Q(x_n)]$$

$$\equiv \exists xP(x) \vee \exists xQ(x)$$

Order Of Nesting Matters

- Is $\forall x\exists y\, P(x,y) \equiv \exists y\forall x\, P(x,y)$ in general?
  - LHS: y can *vary* with respect to x, y is *fixed* with respect to x
  - Let P(x, y) ="x admires y". LHS = "Every person admires some people", RHS= "Some people are admired by everyone"
- Consider $x,y \in \mathbf{R}^+$, and let $P(x,y)$ be xy=1.
  - Is $\forall x\, \exists y\, P(x,y) \equiv \exists y\forall x\, P(x,y)$?
- Consider X={9, 10, 15}, Y={2,3}. Let Q(x,y): y divides x
  - Then, is $\forall x \in X\, \exists y\in Y\, Q(x,y) \equiv \exists y\in Y\, \forall x \in X\, Q(x,y)$?

We have seen so far how quantified statements have truth values, how they can be combined with an AND operator, or negated with a negation operator. We next discuss how they can be combined with the "if then" conditional operator:

$$\boxed{\forall x \in D,\ (P(x) \to Q(x))}$$

which means, for all x in its domain D, if $P(x)$ is true, then $Q(x)$ is true.

Example 32. Take $P(x) =$ "$x > 1$", $Q(x) =$ "$x^2 > 1$", and x has for domain the real numbers $\mathbb{R}$. Then

$$\forall x \in \mathbb{R},\ (P(x) \to Q(x))$$

becomes for all $x \in \mathbb{R}$, if $x > 1$ then $x^2 > 1$.

Attached to the conditional operator were several of its variations, its contrapositive, its inverse, and its negation. We can similarly define these for quantified statements.

$$\begin{array}{ll} \forall x \in D,\ (\neg Q(x) \to \neg P(x)) & \text{contrapositive} \\ \forall x \in D,\ (Q(x) \to P(x)) & \text{converse} \\ \forall x \in D,\ (\neg P(x) \to \neg Q(x)) & \text{inverse} \end{array}$$

See Exercise 25 for a proof that a conditional proposition is equivalent to its contrapositive. See also Exercise 26 to compute the negation of a conditional quantification, namely

$$\neg(\forall\ x, P(x) \to Q(x)).$$

Conditional Quantification (I)

- For all real number x, if $x>1$ then $x^2>1$
  - i.e., any real number greater than 1 has a square larger than 1
- In symbolic form
  - Let *P(x) denote "x>1"*
  - Let *Q(x) denote "$x^2>1$"*
  - Let R denote the domain, the collection of all real numbers
    - $\forall x\ (P(x) \to Q(x))$

Conditional Quantification (III)

- Given a conditional quantification
  - Such as $\forall x \in A\ (P(x) \to Q(x))$
- Then we define

– *contrapositive*	$\forall x \in \mathbf{A}, \neg Q(x) \to \neg P(x)$
– *converse*	$\forall x \in \mathbf{A}, Q(x) \to P(x)$
– *inverse*	$\forall x \in \mathbf{A}, \neg P(x) \to \neg Q(x)$

- Note: A conditional proposition is logically equivalent to its contrapositive

Our motivating Example 21 to start predicate logic was:

Example 33. Consider the following two statements: (1) Every SCE student must study discrete mathematics. (2) Jackson is an SCE student. It looks "logical" to deduce that therefore, Jackson must study discrete mathematics.

We will now develop inference rules, that will allow us to express this example (see Exercise 28).

Consider a predicate variable x taking value in the domain D. Then

$$\boxed{\forall x \in D,\ P(x); \therefore P(c) \text{ for any } c \in D}.$$

As we did for propositional logic, we look at when the premises are true. When $\forall x, P(x)$ is true, $P(x)$ is true for any choice of x in the domain D, in particular it is true for any choice of c, therefore $P(c)$ is true for any $c \in D$. *This rule says that if $P(x)$ is true for any x in a domain, one is allowed to instantiate $P(x)$ in $x = c$ for any choice of $c \in D$.*

Example 34. Suppose we have the following premises: (1) No cat can catch Jerry; (2) Tom is a cat. We want to deduce that therefore Tom cannot catch Jerry. We define two predicates: Cat(x)="x is a cat", Catch(x)="x can catch Jerry". The second premise, Tom is a cat, is the easiest to write, it becomes: Cat(Tom)="Tom is a cat". Now for the first premise, suppose we want to say "cats are catching Jerry", this would be if x is a cat, then x catches Jerry, that is

$$\text{Cat}(x) \to \text{Catch}(x),$$

keeping in mind that we have not yet assigned a quantifier to this statement. To say that "cats are not catching Jerry", then this would be if x is cat, then x cannot catch Jerry, that is

$$\text{Cat}(x) \to \neg\text{Catch}(x),$$

and finally, to say that "no cat can catch Jerry", we add a universal quantifier:

$$\forall x(\text{Cat}(x) \to \neg\text{Catch}(x)).$$

We then have the following premises in predicate logic:

1. $\forall x(\text{Cat}(x) \to \neg\text{Catch}(x))$;

2. Cat(Tom);

We can then instantiate the first premise with x =Tom, to get

$$(\text{Cat}(\text{Tom}) \to \neg\text{Catch}(\text{Tom})).$$

Universal Instantiation

$\forall x\ P(x)$
$\therefore\ P(c)$
where *c* is any element of the domain.

Example:

Cat(x): x is Cat, Catch(x): x can catch Jerry

- No cat can catch Jerry.
- Tom is a cat. Therefore Tom cannot catch

1. $\forall x\ [Cat(x) \rightarrow \neg Catch(x)]$ Hypothesis
2. $Cat(Tom)$ Hypothesis
3. $Cat(Tom) \rightarrow \neg Catch(Tom)$
Universal Instantiation on 1
4. $\neg Catch(Tom)$
Modus ponens on 2 and 3

Universal Generalization

$P(c)$ for any arbitrary c from the domain.
$\therefore\ \forall x\ P(x)$

Example

- P(x) =``x^2 is non-negative” .
- P(c) for an arbitrary real c.
- Therefore P(x) for all x.

But since the second premise says that Cat(Tom); modus ponens ($p \to q$; p; $\therefore q$) tells us that therefore $\neg$Catch(Tom).

Consider a predicate variable x taking value in the domain D. Then

$$\boxed{P(c) \text{ for an arbitrary } c \in D; \therefore \forall x\ P(x) \in D}.$$

This just means that if a predicate is true for an arbitrary element $c \in D$, then it is true for all $x \in D$. Indeed, if whichever premise you look at is true, then the conclusion is true. *This rule allows us to infer $P(x)$ for all $x \in D$ based on $P(c)$ being true for an arbitrary instance $c \in D$.*

Example 35. Consider the premise for an arbitrary real number x, $x^2 \geq 0$. Therefore the square of any real number is non-negative. Set $P(x) =$ "$x^2 \geq 0$". In predicate logic, we have for any arbitrary $c \in \mathbb{R}$, $P(c)$. Therefore $\forall x\ P(x)$.

In fact, we have already used this rule implicitly...In Exercise 2, we showed that if n^2 is even, then n is even, for n an integer. The way we did it, is that we showed the result for one arbitrary n, and concluded this is true for all of them! See Exercise 29 for a more complicated example.

Consider a predicate variable x taking value in the domain D. Then

$$\boxed{\exists x \in D,\ P(x); \therefore P(c) \text{ for some } c \in D}.$$

We look at when the premises are true. When $\exists x, P(x)$ is true, $P(c)$ is true for at least one choice of c in the domain D. *This rule allows to instantiate $P(x)$ in some values of c for which $P(c)$ is true.*

Example 36. The premises are: (1) if any student gets > 80 in the exam, then (s)he gets an A, (2) there are students who get > 80 in the exam, (3) Sam is such a student. We want to conclude that therefore Sam gets an A. Set the predicates $A(x) =$ "x gets an A", $M(x) =$ "x gets > 80 in the exam", the domain D is $D = \{$ students $\}$. In predicate logic, (1) and (2) are respectively given by

1. $\forall x,\ M(x) \to A(x)$.
2. $\exists x,\ M(x)$.

Now that Sam is such a student can be expressed using the above rule, to obtain M(Sam). Then we can instantiate 1. with Sam, to get M(Sam) $\to$ A(Sam), which combined with M(Sam) leads to therefore A(Sam).

Existential Instantiation

$\exists x\, P(x)$
$\therefore$ P(c) for some c in the domain.

Domain={all students}; A(x)="x gets an A"
M(x) = "x gets 80 in the final exam"

Example

- If any student gets >80 in the final exam, then (s)he gets an A.
- There are students who get >80 in the final exam, Sam is such a student.
- Therefore, Sam gets an A.

1. $\forall x\, [M(x) \rightarrow A(x)]$ Hypothesis
2. $\exists x\, M(x)$ Hypothesis
3. M(Sam) Hypothesis +Existential instantiation
4. $M(Sam) \rightarrow A(Sam)$ Universal instantiation on 1
5. A(Sam) Modus ponens on 4 and 3

Existential Generalization

P(c)

$\therefore \exists x\, P(x)$

for c some specific element of domain.

Domain={all people} ,

Sell(x) ="x is selling stocks".

$\forall x\, Sell(x) \rightarrow \exists x\, Sell(x)$

1. $\forall x$ Sell(x) Hypothesis
2. Sell(c) Universal instantiation
3. $\exists x$ Sell(x) Existential generalization

Example

- If everyone is selling stocks, then someone is selling stocks.

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Consider a predicate variable x taking value in the domain D. Then

$$\boxed{P(c) \text{ for some specific } c \in D; \therefore \exists x,\ P(x)}.$$

We look at when the premises are true. When $P(c)$ is true for some specific $c \in D$, $\exists x$ for which $P(x)$ is true. *This rule allows to go from one instantiation $P(c)$ to deduce that there is at least one x for which $P(x)$ is true.*

Example 37. Suppose we want to show formally that if everyone is selling stocks, there must be someone selling stocks. Consider the predicate Sell(x)="x is selling stocks". Then we want to show that if $\forall x$ Sell(x) is true, then $\exists x$ Sell(x) is true as well. From $\forall x$ Sell(x) is true, we instantiate it in one c in the domain (here the domain is { people }). This gives Sell(c). Now using the above rule, we know there must exist at least an x for which Sell(x) is true, as desired.

These 4 rules may look either "obvious" or "contrived", but they are needed to write down things formally, in particular whenever formal methods are involved, e.g., if you need to program formal verifications!

We now come to the last part of predicate logic, namely, we will see how to apply the logic rules we have seen to justify different proof techniques. We will discuss three proof technique: direct proof, induction, proof by contradiction.

Direct Proof. As the name suggests, these are proofs that are perfomed without particular techniques. Some claim has to be shown, and there is a specific way to do so in this particular context.

Example 38. Suppose we want to show that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$, $\forall n \in \mathbb{N}$. Write down the following array:

$$\begin{array}{cccccc} 1 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & n-2 & \dots & 2 & 1 \end{array}$$

If you sum up the entries of the first row, you get $\sum_{i=0}^{n} i$, and if you sum up the entries of the second row, you also get $\sum_{i=0}^{n} i$, Thus if you sum up all entries in this array, you get $2\sum_{i=0}^{n} i$. Now if we sum up the first column, we get $n+1$, if we sum up the second column we get $n+1$, ..., and for the nth column we also get $n+1$, so the total is $n+1$ times n columns.

Proof Techniques

A valid proof is a valid argument, i.e. the conclusion *follows* from the given assumptions.

Three techniques:

- Direct proof
- Proof by induction
- Proof by contradiction.

Proof by example:
The author gives only the case n = 2 and suggests that it contains most of the ideas of the general proof.

Proof by intimidation: 'Trivial.'

Proof by cumbersome notation: Best done with access to at least four alphabets and special symbols.

Proof by exhaustion: An issue or two of a journal devoted to your proof is useful.

Proof by omission: 'The reader may easily supply the details.' 'The other 253 cases are analogous.' '...'

Proof by obfuscation: A long plotless sequence of true and/or meaningless syntactically related statements.

Proof by wishful citation: The author cites the negation, converse, or generalization of a theorem from literature to support his claims.

Proof Technique: Direct Proof

- Prove that $\forall n \in Z, \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- Define $S = \sum_{i=0}^{n} i = \underbrace{0+1+2+\ldots+n-1+n}_{\text{n+1 terms}}$
  - Note: $S = \sum_{i=0}^{n} i = n+n-1+\ldots+2+1+0$
- Sum up: $2S = \overbrace{n+n+\ldots+n+n+n}^{\text{n+1 terms}}$

  $\Rightarrow 2S = (n+1)\,n$
- Thus: $S = \frac{n(n+1)}{2}$

Leonhard Euler (1707-1783)

If we sum up the elements horizontally, we got $2\sum_{i=0}^{n} i$, while if we sum up the elements vertically, we got $n(n+1)$. But the sum does not change when we count differently, thus:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

The legend attributes this proof technique to young Euler, who apparently was punished for not behaving in the classroom. His teacher would have asked him to compute the sum of integers from 1 to 100, and Euler would have, or so the legend says, came up with this technique so as to have to compute all the additions!

Mathematical Induction. This is a proof technique to show statements of the form $\forall n,\ P(n)$. We first explain the technique, then give a proof of why this proof technique is valid, and finally provide an example. A proof by mathematical induction follows two steps:

1. Basis step: You need to show that $P(1)$ is true.
2. Inductive step: You assume that $P(k)$ is true, and have to prove that $P(k+1)$ is then true.

When both steps are complete, we have proved that $\forall n,\ P(n)$ is true. Why is that the case? From the inductive step, we have that

$$P(k) \rightarrow P(k+1)$$

for any k, therefore this is true for when we instantiate in $k=1$, that is

$$P(1) \rightarrow P(2).$$

But from the basis step, we know that $P(1)$ is true, thus combining $(P(1) \rightarrow P(2)); P(1)$; we get that therefore $P(2)$ holds. We can repeat this process with $k=2$ to deduce that $P(3)$ holds, and so on and so forth.

Mathematical Induction

- Prove propositions of the form: $\forall n\ P(n)$
- The proof consists of two steps:
  - Basis Step: The proposition P(1) is shown to be true
  - Inductive Step:
    - Assume P(k) is true (when n=k), then, prove P(k+1) is true (when n=k+1).
- When both steps are complete, we have proved that "$\forall n$ P(n)" is true

Why Does it Work?

- From step 2: $P(1) \rightarrow P(2)$ by Universal Instantiation.
- From step 1: $P(1)$
- Applying *modus ponen*: $P(2)$.
- Repeat the process to get P(3),P(4), P(5), etc. So, all P(k) are true! i.e., $\forall k\ P(k)$

Analogy with climbing Ladders.

Inductive step

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

Basis

Hypothesis

(valid argument)

Example 39. We want to prove

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}, \ \forall n \ \mathbb{N}$$

using mathematical induction. Then set

$$P(n) = \text{``}\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\text{''}, \ \forall n \ \mathbb{N}.$$

- Basis step: $P(1) = 1 = \frac{1(1+1)}{2}$.
- Inductive step: suppose that $P(k)$ is true, that is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ is true for all k. Now we need to show that $P(k+1)$ holds.

$$\begin{aligned}
\sum_{i=0}^{k+1} i &= \sum_{i=0}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.
\end{aligned}$$

There $P(n)$ is true for all n.

Proof by Contradiction. We want to prove that $P(n) \to Q(n)$ is true. In a proof by contradiction, we assume by contradiction that $P(n) \to Q(n)$ is false, that is, that $\neg(P(n) \to Q(n))$ is true. The only way this might happen, is if $P(n)$ is true and $Q(n)$ is false. Thus we start with $P(n)$ true and $Q(n)$ false. If from there we deduce a contradiction, that is a statement of the form $C \wedge \neg C$, which is always false, what we have proven is

$$\neg(P(n) \to Q(n)) \to C \wedge \neg C$$

is true. This is equivalent to $P(n) \to Q(n)$. To see that, set $S(n) = $ "$P(n) \to Q(n)$", and look at the truth table:

S	C	$\neg S$	$C \wedge \neg C$	$(\neg S) \to (C \wedge \neg C)$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Therefore, to prove $P(n) \rightarrow Q(n)$ (or any other statement), it suffices to instead prove the conditional statement $\neg(P(n) \rightarrow Q(n)) \rightarrow C \wedge \neg C$, which is done by direct proof, by assuming $\neg(P(n) \rightarrow Q(n))$ and deduce $C \wedge \neg C$. One difficulty is to figure out what is C given the proof to effectuate.

Example 40. Suppose we want to prove that: if n^2 is even, then n is even, for n integer. Set $P(n)$="n^2 is even", and $Q(n)$="n is even". We want to prove that $P(n) \rightarrow Q(n)$, which is equivalent to $\neg(P(n) \rightarrow Q(n)) \rightarrow C \wedge \neg C$. Suppose $\neg(P(n) \rightarrow Q(n))$, that means $P(n)$ is true and $Q(n)$ is false: n^2 is even, and n is not even (equivalently n is odd). Now if n is odd, then $n = 2k + 1$, and $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, that is n^2 is odd. Thus C = "n^2 is even", and we have just shown that n^2 is odd, that is $C \wedge \neg C$, a contradiction!

We may alternatively use that $P(n) \rightarrow Q(n)$ is equivalent to $\neg Q(n) \rightarrow \neg P(n)$. This would be a proof using contrapositive.

Example 41. Suppose we want to prove that: if n^2 is even, then n is even, for n integer. Set $P(n)$="n^2 is even", and $Q(n)$="n is even". We want to prove that $P(n) \rightarrow Q(n)$, which is equivalent to $\neg Q(n) \rightarrow \neg P(n)$. Suppose that $\neg Q(n)$, that is: n is not even, or n is odd. Now if n is odd, then $n = 2k + 1$, and $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, that is n^2 is odd, which is equivalent to $\neg P(n)$, which concludes the proof.

Example: Mathematical Induction

- Prove that $\forall n \in N, \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- Let P(n) denote $\left[\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right]$

Basis step: P(1) is true

$$1 = \frac{1(1+1)}{2}$$

- Inductive step. Assume P(k) true, k>0: $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$

  Prove P(k+1) true :

$$\sum_{i=0}^{k+1} i = \sum_{i=0}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)[(k+1)+1]}{2}$$

So, *P*(*n*) is true for *n*=*k*+1 and thus true for all n:
$\forall n$ P(n) is true

Proof Technique: Contradiction

- Prove that: If n^2 is even, then n is even, for n integer.
- Lets assume: n^2 is even but n is not even (assume the negation of the statement is true).
- n is not even ⇔ n is odd, i.e., n = 2k+1, k integer.
  - Then $n^2 = (2k+1)^2$
    $= 4k^2 + 4k + 1$
    $= 2(2k^2 + 2k) + 1$ (odd)
- This is in contradiction with the *assumption.*
- Hence, the *assumption is false*, thus the *negation of the assumption is true*.

Exercises for Chapter 3

Exercise 20. Consider the predicates $M(x, y) =$ "x has sent an email to y", and $T(x, y) =$ "x has called y". The predicate variables x, y take values in the domain $D = \{\text{students in the class}\}$. Express these statements using symbolic logic.

1. There are at least two students in the class such that one student has sent the other an email, and the second student has called the first student.
2. There are some students in the class who have emailed everyone.

Exercise 21. Consider the predicate $C(x, y) =$ "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $D = \{\text{courses}\}$. Express each statement by an English sentence.

1. $\exists\, x \in S,\ C(x, \text{MH1812})$.
2. $\exists\, y \in D,\ C(\text{Carol}, y)$.
3. $\exists\, x \in S,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
4. $\exists\, x \in S,\ \exists\, x' \in S,\ \forall y \in D,\ ((x \neq x') \wedge (C(x, y) \leftrightarrow C(x', y)))$.

Exercise 22. Consider the predicate $P(x, y, z) =$ "$xyz = 1$", for $x, y, z \in \mathbb{R}$, $x, y, z > 0$. What are the truth values of these statements? Justify your answer.

1. $\forall\, x,\ \forall\, y,\ \forall\, z,\ P(x, y, z)$.
2. $\exists\, x,\ \exists\, y,\ \exists\, z,\ P(x, y, z)$.
3. $\forall\, x,\ \forall\, y,\ \exists\, z,\ P(x, y, z)$.
4. $\exists\, x,\ \forall\, y,\ \forall\, z,\ P(x, y, z)$.

Exercise 23.

1. Express

$$\neg(\forall\, x,\ \forall\, y,\ P(x, y))$$

in terms of existential quantification.

2. Express

$$\neg(\exists\ x,\ \exists\ y,\ P(x,y))$$

in terms of universal quantification.

Exercise 24. Consider the predicate $C(x,y) =$ "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $C = \{\text{courses}\}$. Form the negation of these statements:

1. $\exists\ x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
2. $\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))$.

Exercise 25. Show that $\forall x \in D,\ P(x) \rightarrow Q(x)$ is equivalent to its contrapositive.

Exercise 26. Show that

$$\neg(\forall\ x, P(x) \rightarrow Q(x)) \equiv \exists x,\ P(x) \wedge \neg Q(x).$$

Exercise 27. Let y, z be positive integers. What is the truth value of "$\exists y,\ \exists z,\ (y = 2z \wedge (y \text{ is prime}))$".

Exercise 28. Write in symbolic logic "Every SCE student studies discrete mathematics. Jackson is an SCE student. Therefore Jackson studies discrete mathematics".

Exercise 29. Here is an optional exercise about universal generalization. Consider the following two premises: (1) for any number x, if $x > 1$ then $x - 1 > 0$, (2) every number in D is greated than 1. Show that therefore, for every number x in D, $x - 1 > 0$.

Exercise 30. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Exercise 31. Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Exercise 32. Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer.