Department of Mathematics and Statistics
University of New Brunswick Fredericton
Math 1003 Intro Calculus I Winter 2021
Final Exam
(35% Toward Final Grade)
April 20, 9am–12noon Atlantic Time
(+30 minutes to upload to Crowdmark)

Work on the test must be your own. You may use the textbook (but you don't need it). You may use the approved online resources (Symbolab, Desmos, WolframAlpha) to help with calculations and to check your results, but not for having the test problems solved for you. If you are using one of these online resources, acknowledge it. Show your work and explain clearly what you are doing in your solutions.

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1. (6 marks) A tanker is leaking oil into the ocean. The oil slick spreads and forms a thin film on the water surface. The surface shape of the spill is a circle with radius r, and it has a constant thickness of $h = \dfrac{1}{1000}$ metres (1 millimetre), so that the volume of the oil slick is
$$V = \frac{\pi r^2}{1000}\,.$$
If the tanker is leaking at the rate of $dV/dt = 100$ cubic metres per hour, how fast is the radius r of the spill increasing when $r = 50$ metres?

[more questions] ⇝

2. (a) (3 marks) Determine whether $f(x)$ has a vertical asymptote at $x = 1$:

$$f(x) = \frac{\sin(x^2 - 1)}{x - 1}$$

(b) (3 marks) Find the horizontal asymptote of this function (defined for $x > 0$):

$$f(x) = \frac{1 + (\ln x)^2}{1 + x^2}$$

[more questions] ⇝

3. In one simplified model of the physical universe, measured distances between points in space are multiplied by a factor that is a function of time:

$$f(t) = a \cdot t^{\frac{2}{3}}$$

where $a > 0$ is some constant and $t > 0$ is time in billions of years since the Big Bang ($t = 0$).

(a) (2 marks) Prove that in this model the universe expands forever: that is, $f(t)$ is increasing for all $t > 0$.

(b) (2 marks) Prove that in this model the rate of expansion, $\frac{df}{dt}$, is decreasing for all $t > 0$.

(c) (1 mark) Since measured distances depend on time, the volume of a spherical lump of matter also changes in time:

$$V(t) = \frac{4\pi}{3} r^3 f(t)^3 = \frac{4\pi}{3} r^3 a^3 t^2,$$

where $r > 0$ is some constant. You are given that $\frac{dV}{dt} = 24\pi r^3 a^3$ at time t_1. Determine t_1.

Note: Your answer in (c) should not involve constants a and r (they should cancel out), and you should get a nice integer value for t_1.

[more questions] ⇝

4. You proved that in the model from Q3 the universe is decelerating (rate of expansion is decreasing). Recent observations indicate that the universe is accelerating its expansion (df/dt is increasing), so the function $f(t)$ in Q3 does not fit the current data. In one of many alternative models, the function $f(t)$ is not given explicitly, but it is assumed that for all $t > 0$ the function $f(t)$ is positive, increasing, and satisfies the acceleration equation

$$f''(t) = -\frac{1}{f(t)^2} - \frac{1+3b}{f(t)^{2+3b}}$$

where b is some constant. Units are adjusted so that $f(t_0) = 1$ at the present time t_0.

(a) (2 marks) Calculate the acceleration at the present time t_0. (Your answer will involve the parameter b.)

(b) (2 marks) Find all values of the parameter b for which the acceleration is positive at the present time.

(c) (2 marks) Assume $b = -\frac{2}{3}$. Prove that $f''(t) = -\frac{1}{f(t)^2} + 1$, and that the universe is accelerating for all $t > t_0$.

Note: This all sounds rather fancy but you can do it. Just focus on the math! In mathematical terms, "acceleration at the present time" simply means $f''(t_0)$. You are given that $f(t_0) = 1$, so put $t = t_0$ in the equation for $f''(t)$ and... take it from there. For part (c), you know that $f(t_0) = 1$ and f is increasing. Use this to prove $f''(t) > 0$ for $t > t_0$.

[Adapted and simplified from: Chevallier, Polarski: "Accelerating universes with scaling dark matter", *Int. J. Mod. Phys. D* **10** (2001) 213–224.]

[more questions] ⇝

5. The Lennard-Jones potential is a simple but important model of attractive and repulsive interactions within a two-atom molecule. It is used in computational chemistry and soft-matter physics. An example of the Lennard-Jones function is given by

$$V(x) = 4\Big(\frac{1}{x^{12}} - \frac{1}{x^6}\Big)$$

where $x > 0$ is the distance between atoms.

(a) (2 marks) Verify that $V'(x) = 24\,\dfrac{x^6 - 2}{x^{13}}$.

(b) (2 marks) Find values of x for which $V(x)$ is increasing, and values of x for which $V(x)$ is decreasing.

(c) (2 marks) Prove that $V(x)$ has a global minimum $V(x_0) = -1$ at $x_0 = \sqrt[6]{2} = 2^{1/6}$. (This x_0 is called the "equilibrium size" of the molecule.)

[more questions] ⇝

6. The position of a point in the plane at time $t \geq 0$ is given by

$$x(t) = t$$
$$y(t) = \sqrt{\frac{2t^3}{3}}$$

(a) (2 marks) Write the function $f(t)$ representing the squared distance of the point $(x(t), y(t))$ from the point $(2, 0)$, and prove that the derivative of $f(t)$ is

$$f'(t) = 2(t-1)(t+2).$$

(b) (2 marks) Find the time t at which the distance is minimal. Use the Second Derivative Test to prove that this is a local minimum.

(c) (2 marks) Find the time t in the closed interval $[0, 2]$ at which the distance is maximal.

Note: Recall that by Pythagoras's theorem the squared distance between two points (x_1, y_1) and (x_2, y_2) in the plane is $(x_2 - x_1)^2 + (y_2 - y_1)^2$.

[end of exam]